Induced linear representations for doubly transitive groups *

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Abstract

This paper concentrates on the classification of irreducible G-modules V in which a collection of 1-dimensional subspaces of V are permuted multiply transitively by G. That is, we consider a group G acting (not necessarily faithfully) on a set Ω, with H the stabiliser of some point $\alpha \in \Omega$, and induce a 1-dimensional (linear) kH-module M to the kG-module M^G, where k is some field.

When M is the trivial kH-module, the reducibility of M^G is known: there is a classical result when $char(k) \nmid |G|$, while the case $char(k) \mid |G|$ has been studied by Mortimer (*Proceedings of the London Mathematical Society*, 41:1–209, 1980). When M is non-trivial, the reducibility of M^G for the families of known doubly transitive groups (and their central extensions) is determined, along with a number of maximal submodules.

The reducibility of M^G is motivated by the classification of primitive affine distance-transitive graphs, since some of these graphs may be constructed from irreducible factor modules of M^G. As such, this paper presents an alternative approach within the classification of primitive affine distance-transitive graphs: we focus on those graphs affording an automorphism group whose stabiliser of the zero vector permutes, in a multiply transitive way, a set of 1-dimensional subspaces of the 'neighbourhood' $\Gamma_1(0)$ of the zero vector.

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1 Introduction

This paper is motivated by the classification of finite primitive distance-transitive graphs, started in [32]. We refer the reader to the introduction of [32] for the basic definitions pertaining to distance-transitive graphs and groups.

Example 1.1 *The Hamming graph $H(d,q)$ has vertex set $V = \Sigma^d$, where Σ is a set of $q \geq 2$ elements, and $x, y \in V$ are adjacent ($x \sim y$) if and only if x and y differ in exactly one coordinate. It has a distance-transitive group of automorphisms given by $Sym(q) \wr Sym(d)$, that is, d copies of the symmetric group $Sym(q)$ each acting on a single coordinate of elements from Σ, and $Sym(d)$ permuting the coordinates.*

Let Γ be a distance-transitive graph with vertex set $V\Gamma$, and $G \leq Aut(\Gamma)$ act distance-transitively on Γ. By a well-known result of Smith [35], imprimitive distance-transitive graphs are either bipartite or antipodal, and there are two simple procedures which allow one to reduce each imprimitive graph to a smaller primitive distance-transitive one. This is the main reason for concentrating initially on the primitive case, as we will do in this paper.

Let Γ, G be as above with valency $k \geq 3$ and diameter $d \geq 2$. The main result of [32] is then that either Γ is a Hamming graph, or G is almost simple, or G is an affine group. In this work will concentrate on the affine case, in which case G has an elementary abelian normal p-subgroup $N \trianglelefteq G$ (with p a prime) acting regularly on $V\Gamma$.

We may interpret G as a subgroup of $AGL(V) = AGL_l(p)$, where $V\Gamma$ is identified with a vector space V of dimension l over $GF(p)$ (i.e. $|N| = p^l$), and $G_0 \leq GL(V)$ is the stabiliser of $0 \in V$. We will write d for the diameter of Γ, equal to the number of orbits of G_0 on $V \setminus \{0\}$.

The classification of the distance-transitive graphs of type (iii) above is greatly reduced by:

Theorem 1.2 (3.2, 3.3, [6]) *Let Γ be as above, with diameter $d \geq 2$, valency $k \geq 3$, on which the affine group G acts primitively and distance-transitively. Then there exists a pair (m, q) with $q^m = |V|$ such that $G_0 \leq \Gamma L_m(q)$, and one of the following holds:*

- (i) *V is a vector space over $GF(q)$ such that the generalised Fitting subgroup $F^*(G_0/Z(G_0)) = L$ is a non-abelian simple group, the projective representation of L on V is absolutely irreducible and can be realised over no proper subfield of $GF(q)$;*
- (ii) *$m = 1$, $q = p^l$ and $G_0 \leq \Gamma L_1(q)$;*
- (iii) *Γ is a Hamming graph, or if $d = 2$ possibly a complement thereof;*

- (*iv*) Γ *is a Bilinear Forms graph, or if* $d = 2$ *possibly a complement thereof; or*
- (*v*) Γ *is an Extra-special graph.*

Cases (*iii*) through (*v*) are well-understood, for example see §1 of [4] and [6]. Case (*ii*) is dealt with in [11].

We therefore concentrate on the affine distance-transitive graphs of type (*i*), where $q = p^a$ for $a \geq 1$ with $am = l$, and $G_0 \leq \Gamma L_m(q)$. By 3.3(*iii*) it then follows that $\Gamma_1(0)$ contains a $GF(q)$-basis of V, so that there is a set

$$S = \{V_1, \ldots, V_s\},$$

of 1-dimensional subspaces in V such that $\Gamma_1(0) = \cup_1^s V_i^{\#}$, where $V_i^{\#} = V_i \backslash \{0\}$ for all i. The elements of S are permuted transitively by G_0, and $V = V_1 + \ldots + V_s$.

To investigate those G_0 with e-transitive actions on S (for $e \geq 2$), we now make the following hypothesis:

Hypothesis 1.3 *Assume* G_0 *acts doubly transitively on* S*.*

By 31.12 [2] we have:

Lemma 1.4 *The generalised Fitting subgroup* $F^*(G_0/Z(G_0)) = L$ *appearing in 1.2 is equal to* $soc(G_0/Z(G_0))$*.*

Furthermore, we may assume $soc(G_0/Z(G_0))$ to be doubly transitive; of the known non-abelian simple doubly transitive groups only the group $P\Sigma L_2(8)$ has a socle which is not doubly transitive in its action on 28 points. Therefore, we may assume G_0 to be a doubly transitive group acting faithfully, or a central extension thereof.

We are now faced with:

Problem 1.5 *With notation as above, and under the previous hypothesis, classify all pairs* (G_0, S)*.*

In case V is a direct sum of the subspaces of S, we call S a *system of imprimitivity* for the G_0-module V and we may appeal to the result (see VII [15] and in particular §12D, §50 there):

Theorem 1.6 *Let* V *be an imprimitive* G_0*-module with a system of imprimitivity* $\{V_1, \ldots, V_n\}$ *on which* G_0 *is transitive. Let* $H \leq G_0$ *be the subgroup consisting of all* $h \in G_0$ *such that* $V_1 h = V_1$*. Then* V_1 *is an* H*-module, and* $V \cong V_1^{G_0}$ *as* G_0*-modules.*

Here $V_1^{G_0}$ is the *induced module*

$$V_1^{G_0} := V_1 \otimes_{KH} KG_0,$$

induced from the stabiliser in G_0 of V_1 to G_0 itself. Note that $V_1^{G_0} \cong V_i^{G_0}$ for all i, where V_i is a H^{x_i}-module for $x_i \in G$ such that $V_i^{x_i} = V_1$.

Recall that $V = V_1 + \ldots + V_s$. If this sum is direct, then Γ is well-known by:

Proposition 1.7 (4.1, [6]) *If* $V = \oplus_{i=1}^{s} V_i$*, the graph* Γ *is a Hamming graph or, if diam*$(\Gamma) = 2$*, the complement of a Hamming graph.*

We will assume otherwise, and concentrate on the pairs (G_0, S) where V is not a direct sum of the subspaces of S. In view of determining which of these modules gives rise to a primitive distance-transitive graph (since no such graph is of course guaranteed), the primitivity of G on Γ is equivalent to the irreducibility of V as a G_0-module (see §3). Thus *we are a led to the interpretation of* V *as an irreducible homomorphic image of* $V_1^{G_0}$*.*

We will call the pair $(G_0, V_1^{G_0})$ *admissable* if $V_1^{G_0}$ is reducible and G_0 acts doubly transitively on the set S of images of V_1 under G_0. We are now faced with the task:

Problem 1.8 *Classify all admissable pairs* $(G_0, V_1^{G_0})$*.*

This is a generalisation of work due to Mortimer [31]. Moreover, we seek to:

Problem 1.9 *Determine the irreducible factor modules of* $V_1^{G_0}$ *for all admissable pairs.*

The work presented here is therefore encompassed by a classification of all irreducible G-modules V in which a collection of 1-dimensional subspaces of V are permuted multiply transitively by G.

We must point out that the distance-transitivity of a possible graph arising from $V_1^{G_0}$ necessitates H mapping $0 \neq x \in V_1$ to $sx \in V_1$, for all $s \in GF(q)^*$, by 3.3. As a consequence, the module $V_1^{G_0}$, with H acting *trivially* on V_1 (leaving all points of V_1 fixed), can only yield a distance-transitive graph if the underlying field is $GF(2)$. Under such circumstances the 2-transitivity of 1-dimensional subspaces is reduced to 2-arc transitivity on vectors, where a *2-arc* is a triplet (x, y, z) of vertices such that $x \sim y, y \sim z$ and $x \neq z$. This problem has been worked out in [23].

2 Statement of Results

Let G be a finite doubly transitive group on Ω such that $G/Z(G)$ acts faithfully, where $|\Omega| = n$. In this case $soc(G/Z(G)) = \overline{N}$ is a non-abelian simple group acting primitively, or an elementary abelian group acting regularly, by 7.2 E in [16].

Suppose $\overline{N}$ is non-abelian simple, and let N be the pre-image of $\overline{N}$ in G. We will assume $H = G_\alpha$ is the point stabiliser and $D = G_{\alpha\beta}$ the two point stabiliser in G of $\alpha, \beta \in \Omega$, taking $t \in G$ to induce $\alpha \leftrightarrow \beta$. Let M be a 1-dimensional (linear) KH-module, where K is a field of characteristic p (possibly $p = 0$), affording the linear representation $\lambda : H \to K$. Then $M^G = M \otimes_{KH} KG$ is a module induced from H to G. When $M = K_H$ is the trivial KH-module, M^G is the permutation module of G on Ω over K, denoted $K\Omega$. The trivial module contained in $K\Omega$ is written $\mathcal{C}$, and we define $\mathcal{D}$ to be the factor module $\mathcal{C}^\perp/\mathcal{C} \cap \mathcal{C}^\perp$ called the *heart* of $K\Omega$.

The main result of this paper is the following, which settles Problem 1.8:

Theorem 2.1 *Suppose K is a field of characteristic $p \neq 0$ such that $p \mid |G|$. Then:*

- (i) *if $M = K_H$ and $p \mid |\Omega|$, we have $K\Omega$ affording no non-trivial irreducible factor module, unless either $\overline{N} = M_{22}$ when $p = 2$, or possibly $\overline{N} = Co_3$ when $p = 3$, with unique non-trivial irreducible factor modules $K\Omega/U$ of dimension listed in Table 2.1;*
- (ii) *if $M = K_H$ and $p \nmid |\Omega|$, we have $K\Omega$ affording $K\Omega/\mathcal{C} \cong \mathcal{C}^\perp$ as the only non-trivial irreducible factor module, unless $\overline{N}$ appears in Table 2.1;*
- (iii) *if $M \neq K_H$ extends to a KG-module $\widehat{M}$ then $M^G \cong \widehat{M} \otimes K\Omega$; and*
- (iv) *if $M \neq K_H$ and M does not extend to G, then M^G is irreducible, unless it appears in in Table 2.2.*

We remark that the second column in both Tables 2.1 and 2.2 lists the degree of the action of G on Ω. The entry $\overline{N} = Co_3$ in Table 2.1 is labelled '(?)' because we have been unable to confirm it. Also, the third column in Table 2.2 provides a necessary condition on M for M^G to be reducible: here $M^* \cong M^{-1}$ is the dual of M, M^q denotes the tensor product of q copies of M, and M_D the restriction to the two point stabiliser D. And finally, the fourth column in Table 2.2 states a restraint on the field while a condition required for the non-trivial M to exist is denoted in brackets.

The proof of 2.1 is a collection of several results. In §4 we discuss first the case $M = K_H$. The reducibility of $\mathcal{D}$ is determined in [31] and charted in Table 4.9, and a series of lemmas from [28] allows us to prove 2.1(i), which appears as 4.6.

$\overline{N}$	Degree n	Condition	Dimension $K\Omega/U$
$PSL_m(q)$	$(q^m-1)/(q-1)$	$q=p^a$ $m \geq 3$	$\binom{p+m-2}{m-1}^a + 1$
M_{22} M_{23}	22 23	$p=2$ $p=2$	10 11
$PSL_2(11)$	11	$p=3$	5
A_7	15	$p=2$	4
HS	176	$p=3$	49
Co_3 (?)	276	$p=3$	126

Table 2.1
Dimensions of the non-trivial irreducible factor modules $K\Omega/U$.

Next, a short list of cases when $p \nmid |\Omega|$ remains, and these are dealt with in subsequent sections where the groups concerned are treated; $PSL_m(q)$ with $m \geq 3$ and $p \mid q$ is dealt with in 7.10, M_{23} with $p=2$ in §8 of [23], $PSL_2(11)$ acting on 11 points with $p=3$ in 7.44, A_7 acting on 15 points with $p=2$ in §7.10, and HS with $p=3$ in §7.11.

In §5 we demonstrate 2.1(iii), which appears as 5.3. We therefore assume M does not extend to G, that is, there is no KG-module $\widehat{M}$ such that $\widehat{M}_H = M$, and determine with 5.9 the condition of reducibility: M^G is reducible whenever $M^t \cong M$ as KD-modules. This is proved in 5.8 by demonstrating that the dimension of $Hom_{KG}(M^G, M^G)$ is at most 2, with equality when said condition holds. Note that, when $p \nmid |G|$, this means M^G has at most two non-isomorphic irreducible direct summands; this is 5.12.

In tackling 2.1(iv), it follows from 5.18 that it is sufficient to work with G simple or finite central extensions thereof: M^G is irreducible if and only if $(M_{N_1})^N$ is irreducible, except in the case where M and $\overline{N}$ are as in Example 5.17. Here $N_1 := H \cap N$, while Example 5.17 finds a unique $M \neq K_H$ for $\overline{N} = PSL_2(q)$ with $q \equiv 1 \pmod 4$ such that $(M_{N_1})^N = U_1 \oplus U_2$ for some U_1, U_2 of equal dimension. Then there exists an $x \in G$ such that $U_1^x \cong U_2$ if and only if M^G is irreducible, by 5.16(ii) and (iii). We remark that these results rely on the assumption that N is itself doubly transitive on Ω, thus excluding the case where $G/Z(G)$ contains $P\Sigma L_2(8)$, which is dealt with separately in §7.9.

For 2.1(iv) it remains to check, for each doubly transitive action (or family thereof), the existence of linear KH-modules and the condition of reducibility in each case. Fortunately, the families $SL_2(q)$, $Sz(q)$, $R(q)$ and $SU_3(q)$ can be handled collectively as examples of groups of Suzuki type, as defined in 6.1. We recite the representation theory of such groups in §6.

X	Degree n	Module M	Condition
A_5	5	M, M^*	$F \geq GF(4)$
$3.A_6$	6	M, M^*	$F \geq GF(4)$
$SL_2(q)$	$q+1$	$M \cong M^*$	$(p \neq 2)$
$SL_m(q)$	$(q^m-1)/(q-1)$	M	$p \mid q$, $m \geq 3$
$Sz(q)$	q^2+1	M	$p \mid q$
$R(q)$	q^3+1	$M \cong M^*$	$(p \neq 2)$
		$M \not\cong M^*$	$p \mid q$
$SU_3(q)$	q^3+1	$M_D \cong M_D^{-q}$	$(q+1 \neq p^b)$
		$M_D \not\cong M_D^{-q}$	$p \mid q$
$Sp_{2m}(2)$	$2^{m-1}(2^m+1)$	M	$(p \neq 2)$
	$2^{m-1}(2^m-1)$	M	$(p \neq 2)$
M_{11}	11	$M \cong M^*$	$p = 3$
$2.M_{12}$	12	$M \cong M^*$	$p = 3$
$P\Sigma L_2(8)$	28	M	$(p^a > 2)$
$3.A_7$	15	M, M^*	$p = 7$
HS	176	$M \cong M^*$	$(p \neq 2)$
$2.HS$	176	$M \cong M^*$	$p = 5$
Co_3	276	$M \cong M^*$	$(p \neq 2)$

Table 2.2
The reducible M^G, where $G \trianglerighteq X$.

The remaining sections 7.1 through 7.12 constitute the calculations and, in particular, make the results of §6 explicit. Throughout these, frequent use is made of the information available in the Atlas [13] and modular atlas [24], as well as a collection of necessary properties for the character λ^G (afforded by M^G) to be reducible or even exist, given in 7.3.

To provide a complete picture, what follows is an overview of the situation over a splitting field K for G of characteristic 0 (see §7 for notation).

Theorem 2.2 *Let λ be a linear K-character of H, where K is a splitting field K for G of characteristic 0. Assume G is simple or a perfect central extension thereof. Then:*

- (i) *if $\lambda = 1_H$, we have $\lambda^G = 1_G + \xi$ as K-characters, where $1_G, \xi$ are irreducible; and*
- (ii) *if $\lambda \neq 1_H$, then the possible K-characters λ^G are included in Tables 2.3*

G	$\lvert G_1\rvert$	Degree n	Condition	No. of characters for G	Degrees in the decomposition
A_5	$\lvert A_4\rvert$	5		1	$5a$
$3.A_6$	$3\lvert A_5\rvert$	6		2	$6a, \overline{6a}$
$SL_2(q)$	$\lvert GF(q)^*\rvert$	$q+1$	q even	$\frac{1}{2}(q-2)$	$(q+1)$
			q odd	1	$(\frac{1}{2}(q+1)) + (\frac{1}{2}(q+1))$
				$\frac{1}{2}(q-3)$	$(q+1)$
$SL_m(q)$	$\lvert GL_{m-1}(q)\rvert q^{m-1}$	$q^m - 1/q - 1$		$\lvert GF(q)^*\rvert - 1$	$(q^m - 1/q - 1)$
$SU_3(q)$	$q^3(q^2-1)$	q^3+1		$q-2$	$(q^2-q+1) + (q(q^2-q+1))$
				$\frac{1}{2}(q-2)(q+1)$	(q^3+1)
$Sp_{2m}(2)$	$\lvert O^+_{2m}(2)\rvert$	$2^{m-1}(2^m+1)$		1	$(\frac{1}{3}(2^{2m}-1)) + (\frac{1}{3}(2^m+1)(2^{m-1}+1))$
	$\lvert O^-_{2m}(2)\rvert$	$2^{m-1}(2^m-1)$		1	$(\frac{1}{3}(2^{2m}-1)) + (\frac{1}{3}(2^m-1)(2^{m-1}-1))$
$Sz(q)$	$q^2(q-1)$	(q^2+1)		$\frac{1}{2}(q-2)$	(q^2+1)
$R(q)$	$q^3(q-1)$	(q^3+1)		1	$(q^2-q+1) + (q(q^2-q+1))$
				$\frac{1}{2}(q-3)$	(q^3+1)

Table 2.3. The decomposition of λ^G for the infinite families of doubly transitive groups G, with G simple or a perfect central extension thereof.

G	G_1	Degree n	Decomposition
M_{11}	$M_{10} \cong A_6 \cdot 2$	11	$1a + 10a$
			$11a$
M_{12}	M_{11}	12 (twice)	$1a + 11a, 11b$
$2.M_{12}$	$2 \times M_{11}$		$12b$
M_{22}	$PSL_3(4)$	22	$1a + 21a$
M_{23}	M_{22}	23	$1a + 22a$
M_{24}	M_{23}	24	$1a + 23a$
M_{11}	$PSL_2(11)$	12	$1a + 11a$
$PSL_2(11)$	A_5	11 (twice)	$1a + 10b$
$P\Sigma L_2(8)$	$9{:}6$	28	$1a, 1b, 1c + 27a$
			$7a, 7b, 7c + 21a$
A_7	$PSL_2(7)$	15 (twice)	$1a + 14b$
$3.A_7$	$3 \times PSL_2(7)$		$15b, \overline{15b}$
HS	$PSU_3(5) : 2$	176 (twice)	$1a + 175a$
			$22a + 154b, 154c$
$2.HS$	$2 \times (PSU_3(5) : 2)$		$176a, 176b$
Co_3	$McL : 2$	276	$1a + 275a$
			$23a + 253b$

Table 2.4
The decomposition of induced linear characters for 'exceptional' doubly transitive groups G, with G simple or a perfect central extension thereof.

and 2.4.

A number of cases of Problem 1.9 have also been worked out:

Theorem 2.3 *The confirmed dimensions of the irreducible factor modules* M^G/U, *with* $M \neq F_H$, *are listed in Table 2.5. These include all admissable pairs* (G, M^G) *except those cases with* G *containing*

$$SL_2(q), Sz(q), R(q) \text{ or } SU_3(q)$$

and $p \mid q$, *or* $G = SU_3(q)$ *and* $p \mid q + 1$, *or* $G = Sp_{2m}(2)$ *with* $p \neq 2$.

Note that the entry $SL_2(q)$ in Table 2.5 is labelled '(!)' as M^G may be irreducible (with $G \trianglerighteq SL_2(q)$) as mentioned in the remarks concerning 2.1(iv) above.

X	Module M	Condition	Dimension M^G/U
A_5	M, M^*	$F \geq GF(4)$	$2, 2^*$
$3.A_6$	M, M^*	$F \geq GF(4)$	3
$SL_2(q)$ (!)	$M \cong M^*$	$p \nmid q$	$\frac{1}{2}(q+1)$, $\frac{1}{2}(q+1)^*$
$R(q)$	$M \cong M^*$	$p \mid (q-1)(q^2-q+1)$	(q^2-q+1), $q(q^2-q+1)$
		$p \mid q+1$	q^2-q+1
$SU_3(q)$	$M_D \cong M_D^{-q}$	$3 \neq p \mid (q^2-q+1)$	(q^2-q+1), $q(q^2-q+1)$
		$2 \neq p \mid (q-1)$	(q^2-q+1), $q(q^2-q+1)$
M_{11}	$M \cong M^*$	$p = 3$	5
$2.M_{12}$	$M \cong M^*$	$p = 3$	6
$P\Sigma L_2(8)$	$M^3 \cong F$	$F \geq GF(q)$, $q = 4, 7$	1
		$3 \mid q-1$, $q \neq 4, 7$	1, 27
	$M^3 \ncong F$	$2 \nmid p$	7, 21
$3.A_7$	M, M^*	$p = 7$	6, 9
HS	$M \cong M^*$	$p \neq 5$	22, 154
		$p = 5$	21
$2.HS$	$M \cong M^*$	$p = 5$	28
Co_3	$M \cong M^*$	$p = 3$	22
		$p = 5$	23
		$p \neq 3, 5$	23, 253

Table 2.5
Confirmed dimensions of the irreducible factor modules M^G/U, where $G \trianglerighteq X$.

The ‘exceptional’ cases in 2.3 follow, as above, from an inspection of the ATLAS [13], modular atlas [24] and use of 7.3, while the groups of Suzuki type $SL_2(q)$ are handled in 7.12, 7.13, $R(q)$ in §7.4, and $SU_3(q)$ in §7.5.

The admissable pairs that give rise to distance-transitive graphs are included in [30], [7] and [12], dealing in turn with the situation in which the non-abelian simple group L appearing in 1.2(i) is alternating, sporadic or a group of Lie type embedded cross characteristically. Checking these results against our own we distil that:

Lemma 2.4 *With notation as in §1, the admissable pairs* $(G_0, V_1^{G_0})$ *supporting the structure of a primitive distance-transitive graph are listed in Table 2.6.*

$F^*(G_0/Z(G_0))$	$q = p^a$	$dim(V(q))$	G_0-orbits
A_5	4	2	$1, 15$
A_6	4	3	$1, 18, 45$
$PSL_2(7)$	9	4	unitary
$PSL_2(11)$	3	6	(see $2.M_{12}$)
$PSL_2(13)$	3	7	quadratic
$Sp_6(2)$	$3, 5, 7, 11, 13$	7	quadratic
$SU_3(3)$	5	7	quadratic
M_{11}	3	5	$1, 22, 220$
M_{12}	3	6	$1, 24, 264, 440$
M_{22}	2	10	$1, 22, 231, 770$
M_{23}	2	11	$1, 23, 253, 1771$

Table 2.6
Known distance-transitive graphs with $V = V_1^{G_0}/U$.

In the last column of Table 2.6 we list the G_0 orbits on V, and write 'quadratic' or 'unitary' if G_0 preserves a non-degenerate form on V of such a type (up to scalar multiplication and field automorphisms). By the following result due to van Bon [5], the distance-transitive graphs in this case are well-known:

Lemma 2.5 *Let G be a primitive affine distance-transitive automorphism group of Γ, identify $V \trianglelefteq G$ with Γ and suppose that V carries a $GF(q)$-structure preserved by G_0 as before. If G_0 contains the group of scalars $GF(q)^*$ and preserves a non-degenerate quadratic form on V, up to scalar multiplication and field automorphisms, then either $d \leq 2$, Γ is a Hamming graph, a half cube, a folded cube or a folded half cube.*

The above lemma includes the case of invariant unitary forms since these may be considered as orthogonal forms over the prime subfield, as noted in [7]. Here d is the diameter of Γ, and the half cube, folded cube and folded half cube are described, for example, in Chapter 9D [8].

Further results deal with the case G acting doubly transitively (though not necessarily faithfully) on Ω such that $soc(G/\operatorname{Ker} G)$ is elementary abelian and regular; these include the so-called Frobenius groups. In 5.5 we find that, provided G has a regular normal subgroup N, then $M^G \cong \widehat{M} \otimes K\Omega$, where $K\Omega$ is the permutation module of G on Ω. The reducibility of $\mathcal{D}$ in this case is also well-known by 5.6.

Lastly we point out that often we do not require G to act faithfully nor that $\operatorname{Ker} G \leq Z(G)$ (i.e. that the extension is central). In the places where these assumptions are made care has been taken to make them explicit. However, all the calculations in §7.1 through §7.12 concentrate on G being a simple doubly transitive group, or a finite central extension thereof, in view of the remarks on 2.1(*iv*) above.

3 Preliminary Results

We adopt the notation introduced in §1. For Γ connected, distance-transitive of diameter $d \geq 2$, the *distance* between $\alpha, \beta \in V\Gamma$ is the length of the shortest path between those two vertices, denoted $\delta(\alpha, \beta)$. We put $\Gamma_i(\alpha) := \{\beta \in V\Gamma : \delta(\alpha, \beta) = i\}$, for $0 \leq i \leq d$. If $\delta(\alpha, \beta) = i$ (for some $0 \leq i \leq d$), denote by c_i, a_i, b_i the number of vertices adjacent to β, at distances $i-1$, i and $i+1$ from α, respectively. (Here c_0, b_d are taken to be zero.) We will write $k_i := |\Gamma_i(\alpha)|$.

The following summarises some of the basic restrictions on the constants of the intersection array.

Lemma 3.1 (4.1.6, 5.1.1, [8]) *Suppose* Γ *is a primitive distance-transitive graph with valency* $k \geq 3$ *and diameter* $d \geq 3$*. Then:*

(*i*) $k_1 > b_1 \geq \ldots \geq b_{d-1}$ *and* $1 = c_1 \leq c_2 \leq \ldots \leq c_d$*;*
(*ii*) *there exist integers* h, l *with* $h \leq l \leq d$ *such that*

$$1 < k_1 < \ldots < k_h = \ldots = k_l > \ldots > k_d,$$

and in particular, $k_i \leq k_j$ *for* $0 \leq i \leq j$, $i + j \leq d$*;*
(*iii*) *if* $k_i = k_{i+1}$ *then* $k_i \geq k_j$ *for all* j*; and*
(*iv*) *if* $k_{j-1} = k$ $(3 \leq j \leq d)$*, then either* Γ *is a polygon* $(k = 2)$ *or* Γ *is antipodal with* $j = d$ *and* $k_d = 1$*.*

If G acts on N, then $U \leq N$ is said to be G*-invariant* if $U^x = U$ for all $x \in G$. Moreover, we say G *acts irreducibly* if the only G-invariant subgroups of N are trivial. The assumption that G acts primitively on $V\Gamma = N$ is equivalent to G_0 acting irreducibly on N, by the following result:

Lemma 3.2 *Let* G *be a group,* $N, H \leq G$ *subgroups with* N *normal in* G*,* $N \cap H = 1$ *and* $G = NH$*. Then the following are equivalent:*

(*i*) H *acts irreducibly on* N*; and*
(*ii*) H *is maximal in* G*.*

In particular, if (i) or (ii) holds then $N_G(H) = H$*.*

We now adopt the setting and notation of 1.2. So G is primitive on $V\Gamma$, identified with V a G_0-module over $GF(q)$ ($|V| = p^l = q^m$) where $d \geq 2$, $k \geq 3$.

Lemma 3.3 *Let G_0, Γ and V be as in 1.2. Then:*

(*i*) $\Gamma_1(0)$ *contains a $GF(p)$-basis of V;*
(*ii*) *if $x \in \Gamma_1(0)$ and $s \in GF(p)^*$, we have $sx \in \Gamma_1(0)$; and*
(*iii*) *if $x \in \Gamma_1(0)$ and $s \in GF(q)^*$, we have $sx \in \Gamma_1(0)$.*

PROOF. (*i*) Because G is primitive on $V\Gamma$, V is an irreducible G_0-module. The $GF(p)$-span of $\Gamma_1(0)$ is a G_0-module, and as Γ is connected, this must be V.

(*ii*) Let E be the set of $r \in GF(p)^*$ such that $rx \in \Gamma_1(0)$, and suppose that $sx \notin \Gamma_1(0)$, for some $s \in GF(p)^*$. Then Es is the coset of E in $GF(p)^*$ containing s, and there exists a G_0-orbit $\Gamma_j(0)$ say, such that $rx \in \Gamma_j(0)$ for all $r \in Es$. By distance-transitivity we now have $k_j = |\Gamma_j(0)| = |\Gamma_1(0)| = k$. But then either Γ is a polygon or Γ is antipodal, by 3.1. By primitivity of G, Γ is not antipodal, and Γ is not a polygon ($k \neq 2$) by hypothesis, so the result follows.

(*iii*) This is analogous to (*ii*), using instead the coset E of $GF(p)$ in $GF(q)$ containing s.

By 3.3(*iii*), $\Gamma_1(0)$ contains a $GF(q)$-basis of V. Now the neighbourhood $\Gamma_1(0)$ of 0 may be written $\Gamma_1(0) = \cup_{i=1}^{s} V_i^{\#}$, where $V_i^{\#} = V_i \setminus \{0\}$ for all i and V_i appears in a collection

$$S = \{V_1, \ldots, V_s\},$$

of 1-dimensional subspaces over $GF(q)$. Here the elements of S are permuted transitively by G_0, and $V = V_1 + \ldots + V_s$. If this sum is direct, then Γ is well-known by 1.7.

Let $H \leq G_0$ be the stabiliser of V_1. Then V_1 is a 1-dimensional (or *linear*) KH-module, with $K = GF(q)$. By 1.6 we now seek irreducible factor modules of the induced module $V \cong (V_1)^{G_0}$ (defined in §1).

Let $V_1 = Ke$ for some basis vector e. Then we may define the *linear representation* $\lambda : H \to K$ by $ex = \lambda(x)e$ where $x \in H$. It is easy to see that $H' \leq \operatorname{Ker}\lambda$, where H' is the commutator subgroup of H. Consequently, λ induces $\tilde{\lambda} : H/H' \to K$, given by $\tilde{\lambda}(H'x) = \lambda(x)$ for all $x \in H$. Conversely, we may define a linear representation λ of H given $\tilde{\lambda}$ of H/H'. In general:

Lemma 3.4 *The linear modules of G are lifted from G/G'.*

For a permutation group G acting (not necessarily faithfully) on Ω, the *permutation module* $K\Omega$ over K is the set of all functions $f : \Omega \to K$ under point-wise addition and scalar multiplication by elements of K, that is, $(f_1 + f_2)(\alpha) = f_1(\alpha) + f_2(\alpha)$ and $(\mu f_1)(\alpha) = \mu f_1(\alpha)$ for all $\alpha \in \Omega$, $\mu \in K$. The action of G on the function f is given by $f^x(\alpha) := f(\alpha^{x^{-1}})$, for all $\alpha \in \Omega$, $x \in G$. As a standard basis we choose the *characteristic functions* of Ω, that is, the set of χ_α such that $\chi_\alpha(\beta) = 1$ if $\beta = \alpha$ and 0 otherwise $(\alpha, \beta \in \Omega)$. We will often refer to the set Ω of size n as the set $\{1, 2, \ldots, n\}$, implying an arbitrary but fixed ordering for convenience of notation alone.

It follows that $K\Omega$ is a direct sum of the subspaces spanned by the χ_i for $1 \leq i \leq n$, and so if G acts transitively on Ω, then $K\Omega \cong (K\chi_i)^G$ by 1.6. For example, when V_1 above is the trivial KH-module, we have $(V_1)^{G_0}$ equal to the permutation module over $GF(q)$ of G_0 acting on S. Note that G need not act faithfully, since $\operatorname{Ker} G \leq H$ in any transitive action will act trivially on K_H, and the lifting of modules 'commutes' with induction.

In view of Hypothesis 1.3, lists of the known doubly transitive groups can be found in §5 of [10], or [26], and these lists are complete in as far as the classification of finite simple groups is complete. In this paper we will work with currently known doubly transitive groups, as presented in in §5 of [10]. The Schur multipliers of the finite doubly transitive groups are listed in Table 3.7, unless they do not follow the general pattern, in which case they appear in Table 3.8.

4 The Permutation Module, $K\Omega$

If G acts transitively on Ω, then it easy to see $K\Omega$ contains a unique trivial submodule, denoted $\mathcal{C}$, namely the subspace of constant functions. Let $(-,-) : K\Omega \times K\Omega \to K$ be the inner product given by $(f, g) := \sum_{\gamma \in \Omega} f(\gamma)g(\gamma)$, where $f, g \in K\Omega$. It is easily established that this form is bilinear, symmetric and G-invariant, and with respect to it, every submodule of $K\Omega$ has an associated submodule to which it is said to be *orthogonal*: if L be a submodule of $K\Omega$, then so is $L^\perp := \{f \in K\Omega : (f, g) = 0 \text{ for all } g \in L\}$. In particular, $\mathcal{C}^\perp$ can be characterised as the submodule of functions f such that $\sum_{\gamma \in \Omega} f(\gamma) = 0$, and we usually take $\{\chi_\alpha - \chi_\beta : \beta \in \Omega\}$ to be basis for this submodule.

Now $\mathcal{C} \leq \mathcal{C}^\perp$ if and only if $p \mid |\Omega|$, and we may define [31]:

Definition 4.1 *The heart over the field K of the group G acting on Ω is the G-module*

$$\mathcal{D} := \mathcal{C}^\perp / \mathcal{C} \cap \mathcal{C}^\perp.$$

G	$\lvert M(G)\rvert$	Extension
A_m	2	$2.A_m$
$PSL_m(q)$	$(m, q-1)$	$SL_m(q)$
$PSU_3(q)$	$(3, q+1)$	$SU_3(q)$
$Sp_{2m}(2)$	1	-
$Sz(q)$	1	-
$R(q)$	1	-
M_{11}	1	-
M_{12}	2	$2.M_{12}$
M_{22}	12	$12.M_{22}$
M_{23}	1	-
M_{24}	1	-
HS	2	$2.HS$
Co_3	1	-

Table 3.7
The Schur multipliers of the finite doubly transitive groups.

G	$M(G)$
A_6, A_7	$\mathbb{Z}_6$
$PSL_2(4) \cong SL_2(4), PSL_3(2) \cong SL_3(2),$ $PSL_4(2) \cong SL_4(2), Sp_6(2)$	$\mathbb{Z}_2$
$PSL_2(9) \cong A_6$	$\mathbb{Z}_6$
$PSL_3(4)$	$\mathbb{Z}_4 \times \mathbb{Z}_{12}$
$Sz(8)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$

Table 3.8
Exceptional Schur multipliers of finite doubly transitive groups.

We know that $\mathcal{C}, \mathcal{C}^{\perp}$ are always submodules of $K\Omega$, with $\mathcal{C}$ simple. Moreover:

Lemma 4.2 (2, [31]) *If G is a transitive permutation group on a set Ω, and the heart over K of G on Ω is simple, then the only G-invariant subspaces of $K\Omega$ are $0, \mathcal{C}, \mathcal{C}^{\perp}, K\Omega$.*

We are interested in determining the irreducible homomorphic images of $K\Omega$ in view of Problem 1.9:

Problem 4.3 *Classify all pairs $(G, \mathcal{D})$ such that $\mathcal{D}$ is reducible.*

G	Degree n	Condition
$G \leq A\Gamma L_m(q)$	q^m	$p \mid q$
$PSL_2(q)$ $q \equiv \pm 1 \pmod 8$	$q+1$	$K \geq GF(2)$
$PSL_2(q)$ $q \equiv \pm 1 \pmod 4$	$q+1$	$K \geq GF(4)$
$PSL_m(q)$ $m \geq 3$	$q^m - 1/q - 1$	$p \mid q$
$PSU_3(q)$	q^3+1	$p \mid q+1$
$Sp_{2m}(2)$	$2^{m-1}(2^m \pm 1)$	$p = 2$
$Sz(q)$	q^2+1	$p \mid q+1+r$ $r^2 = 2q$
$Re(q)$	q^3+1	$p \mid (q+1)(q+1+r)$ $r^2 = 3q$
M_{22} M_{23} M_{24}	22 23 24	$p = 2$ $p = 2$ $p = 2$
M_{11}	12	$p = 3$
$PSL_2(11)$	11	$p = 3$
A_7	15	$p = 2$
HS	176	$p = 2, 3$
Co_3	276	$p = 2, 3$

Table 4.9
The reducibility of the heart $\mathcal{D}$

This is the question taken up by Mortimer in [31], where the reducibility of the heart $\mathcal{D}$ is investigated for all doubly transitive groups, with two small exceptions; the case $G = Co_3$ for $p = 2, 3$, and the case $G = R(q)$ for $p > 3$ are both discussed in 0.5 [9]. The results are listed in Table 4.9.

One factor module of $K\Omega$ is always $\mathcal{C} \cong K\Omega/\mathcal{C}^{\perp}$, but clearly this is not the vertex set of a distance-transitive graph. When $p \nmid n = |\Omega|$ we have $K\Omega = \mathcal{C} \oplus \mathcal{C}^{\perp}$, and we seek factor modules of $\mathcal{C}^{\perp}$. When $p \mid n$ we have:

Lemma 4.4 (7(a), [28]) *Let $U < K\Omega$ be a proper submodule. If $n = p^a$, for some integer $a \geq 1$, then $\mathcal{C} \leq U \leq \mathcal{C}^{\perp}$.*

Lemma 4.5 (7(c), [28]) *Let $U < K\Omega$ be a proper submodule. If $p \mid n$ and G_1 contains a subgroup L transitive on $\Omega \setminus \{1\}$, such that $p \nmid |L \cap G_{12}|$, then $\mathcal{C} \leq U \leq \mathcal{C}^{\perp}$.*

In particular, this is true for all groups of Suzuki type, since $L \cap G_{12} = 1$ (see §6).

Lemma 4.6 *Suppose G acts doubly transitively on Ω, with $\operatorname{Ker} G$ the kernel in this action. If $p \mid n$ then $soc(K\Omega) = \mathcal{C}$, except when $p = 2$ and $G/\operatorname{Ker} G \trianglerighteq M_{22}$ acts on 22 points, and possibly when $p = 3$ and $G/\operatorname{Ker} G = Co_3$.*

PROOF. Since the lifting of modules 'commutes' with induction, we may assume G to be faithful, that is, we may work with $G/\operatorname{Ker} G$ and lift our results back to G. When $p \mid n$ we have $\mathcal{C} \leq \mathcal{C}^{\perp}$, and the result follows if $\mathcal{D}$ is irreducible, by 4.2. If $\mathcal{D}$ is reducible then it appears in Table 4.9. We find that if $G \leq A\Gamma L_m(q)$, the result follows by 4.4. If the group is of Suzuki type, then the result is true by 4.5. When $p = 2$, the results for the sporadic groups and $Sp_{2m}(2)$ can be found in [23]. This leaves M_{11} and Co_3, for $p = 3$. In the former case, the point stabiliser $H = PSL_2(11)$ contains a cyclic $P \in Syl_{11}(H)$ which is regular on the remaining 11 points, and the result follows from 4.5. We have been unable to rule out the latter case.

The natural permutation module over $GF(2)$ for the group M_{22} does indeed give rise to a primitive distance-transitive graph, as listed in Table 2.6 (see 1.8(d) [23]).

In view of Problem 1.9 with the trivial module $V_1 \cong K$ (notation as in §1), we therefore seek only those reducible hearts with $p \nmid |\Omega|$, so that by Table 4.9 it remains to consider the cases when $soc(G/\operatorname{Ker} G)$ is the group:

(*i*) $PSL_m(q)$, $m \geq 3$, with $p \mid q$;
(*ii*) M_{23} with $p = 2$;
(*iii*) $PSL_2(11)$ acting on 11 points, with $p = 3$;
(*iv*) A_7 acting on 15 points, with $p = 2$; and
(*v*) HS with $p = 3$.

These will be dealt with in the subsequent sections, and include the cases $K = GF(2)$ that have already been tackled in [23].

5 The Monomial Module, M^G

We will adopt the following notation as standard, unless stated otherwise.

Hypothesis 5.1 *Assume that:*

- (i) *G acts (not necessarily faithfully) doubly transitively on Ω where $|\Omega| = n$;*
- (ii) *$H = G_1$, the point stabiliser in G of $1 \in \Omega$;*
- (iii) *M is a 1-dimensional non-trivial KH-module; where*
- (iv) *K is a field of characteristic p (possibly $p = 0$);*
- (v) *$\lambda : H \to K$ is the linear representation afforded by M;*
- (vi) *ν is the matrix representation afforded by M^G (i.e. $\nu = \lambda^G$); and*
- (vii) *t is an element such that $1^t = 2$.*

We choose a natural basis for $M^G = M \otimes_{KH} KG$, one in keeping of the natural system of imprimitivity of this induced module, namely

$$\{e_i := e \otimes g_i : 1 \leq i \leq n\}, \tag{5.1}$$

where $M = Ke$ and the g_i form a right transversal for H in G. Here $n = |\Omega| = |G : H|$ since G acts transitively on Ω. We set $g_1 = 1$, $g_2 = t$. With respect to this basis, the matrix representation ν afforded by M^G are called *monomial matrices*: for every $x \in G$, $\nu(x)$ has an element of $\operatorname{Im} \lambda$ in each row and column and zeros everywhere else.

We say λ *extends* to G if there exists some homomorphism $\hat{\lambda} : G \to K$ whose restriction to H is λ.

Proposition 5.2 *The linear representations of $H \leq G$ extend to G if and only if $H' = G' \cap H$.*

PROOF. If μ is a linear representation of G, then μ_H is a linear representation of H. Therefore, $H' \geq G' \cap H$ when all linear representations of H extend to G. But $H' \leq G'$, so we require $H' = G' \cap H$. Conversely, when $H' = G' \cap H$ we have

$$H/H' \cong H/(G' \cap H) \cong HG'/G'.$$

Now, the linear representations of HG'/G' extend to G/G'. Therefore, given a linear representation λ of H/H' lifted to H, there exists a linear representation μ lifted from HG'/G' to HG', such that $\mu_H = \lambda$ and λ extends to G.

For the next result, suppose $H \leq G$ are arbitrary groups, and L a KH-module over an arbitrary field K.

Theorem 5.3 *Let $H \leq G$ be groups and $\Omega = G/H$ be a set of right cosets. Suppose L is a KH-module over a field K, extendible to $\widehat{L}$, a KG-module. Then $L^G \cong \widehat{L} \otimes K\Omega$, where $K\Omega$ is the permutation module of G on Ω.*

PROOF. Since G acts transitively on Ω, we may conclude that $K\Omega \cong (K_H)^G$. But $\widehat{L} \otimes (K_H)^G \cong (\widehat{L}_H \otimes K_H)^G$, since in general $(U_H \otimes V)^G \cong U \otimes V^G$ for arbitrary U, V respectively H- and G-modules, while $U \otimes K \cong U$.

Therefore the study of the structure of L^G is reduced to that of the permutation module $K\Omega$ in the case L extends to a KG-module. For example:

Lemma 5.4 *Let* $G = NH$ *be groups such that* $N \lhd G$ *and* $N \cap H = 1$*. If* L *is any* KH*-module, then* L *extends uniquely to a* KG*-module* $\widehat{L}$ *such that* $\widehat{L}_H = L$*.*

In particular, when G is a doubly transitive group with regular normal socle, then $G = NH$ because N is transitive and $N \cap H = 1$ because N is regular, so:

Corollary 5.5 *Suppose that* G *has a regular normal subgroup* N*. Then*

$$M^G \cong \widehat{M} \otimes K\Omega,$$

where $K\Omega$ *is the permutation module of* G *on* Ω*.*

Because of 5.5 it is natural to enquire about the reducibility of the heart of G over K on Ω. Let $p = char(K)$ as before.

Theorem 5.6 (§3, [31]) *Let* $G \leq AGL_m(q)$ *(for* q *a prime) and* $\mathcal{D}$ *be the heart over* K *for* G*. Then* $\mathcal{D}$ *is irreducible if and only if* $p \neq q$*, provided* $(m, q) \neq (1, 2), (1, 3)$ *or* $(2, 2)$*.*

Unless stated otherwise, we will assume M does not extend to a KG-module.

Lemma 5.7 *If* G *is doubly transitive on* Ω*, then it has two* (G_α, G_β)*-double cosets, where* G_α, G_β *are the point stabilisers of* $\alpha, \beta \in \Omega$ *respectively, not necessarily* $\alpha \neq \beta$*.*

We will write the (H, H)-double coset decomposition of G as $G = H1H \cup HtH$, where t is the element $1^t = 2$. Moreover, $H^t = t^{-1}Ht$ and set $D := H^t \cap H$ be the two point stabiliser G_{12} in G. Also, let $i_{KG}(U, V) = dim(Hom_{KG}(U, V))$.

Theorem 5.8 *Let* M *be a 1-dimensional* KH*-module,* L *a 1-dimensional* KH^t*-module, where* K *is an arbitrary field and* t *is as before. Then:*

(i) $i_{KG}(M^G, L^G) = i_{KD}(M, L) + i_{KH^t}(M^t, L)$*; and*
(ii) $i_{KG}(M^G, M^G) = i_{KD}(M^t, M) + 1$*.*

In particular, $i_{KG}((M_H)^G, (M^t_{H^t})^G) = i_{KD}(M, M^t) + 1$*.*

PROOF. From Frobenius Reciprocity we learn

$$Hom_{KG}(M^G, L^G) \cong Hom_{KH^t}((M^G)_{H^t}, L),$$

while Mackey's Theorem establishes

$$\begin{aligned}(M^G)_{H^t} &\cong (M^1_{H^1 \cap H^t})^{H^t} \oplus (M^t_{H^t \cap H^t})^{H^t} \\ &\cong (M_D)^{H^t} \oplus M^t,\end{aligned}$$

by using 5.7. Substituting, we have

$$\begin{aligned}Hom_{KG}(M^G, L^G) &\cong Hom_{KH^t}((M_D)^{H^t}, L) \oplus Hom_{KH^t}(M^t, L) \\ &\cong Hom_{KD}(M, L) \oplus Hom_{KH^t}(M^t, L),\end{aligned}$$

by using Frobenius Reciprocity for the first direct summand. This proves (i), and (ii) is proved similarly.

Corollary 5.9 *If* $M^t \cong M$ *as* KD*-modules, then* M^G *is reducible.*

Corollary 5.10 *If* $M^t \not\cong M$ *as* KD*-modules, then* M^G *is indecomposable.*

Corollary 5.11 *The module* $(K_H)^G$ *is indecomposable if and only if* $p \mid |\Omega|$.

Corollary 5.12 *If* $p \nmid |G|$, *then* M^G *is reducible if and only if* $M^t \cong M$ *as* KD*-modules, in which case* M^G *has precisely two non-isomorphic, simple direct summands.*

Lemma 5.13 *Suppose* $U \leq M^G$ *is a non-zero proper submodule. Then* $U_H \leq V$, *for a submodule* $V \cong (M^t_D)^H$, *and* V *contains* M *as a composition factor.*

PROOF. We adopt the standard basis (5.1) for M^G, where $M = Ke$ and $e_1 = e \otimes g_1$ for the identity $g_1 \in H$. Then $(M^G)_H = Ke_1 \oplus V$, where $V \cong (M^t_D)^H$, by Mackey's Theorem and the Krull-Schmidt Theorem. If $U \leq M^G$ is a non-zero submodule, then $U \geq Ke_1$ implies $U = M^G$ by the transitivity of G on the standard basis. But U_H has a homomorphic image onto M by Frobenius Reciprocity, so the result follows.

We now turn to those doubly transitive groups G for which $soc(G/\operatorname{Ker} G) = \overline{N}$ is non-abelian simple and primitive on the set underpinning the group action. Then $G/\operatorname{Ker} G$ occurs as $\overline{N} \leq G/\operatorname{Ker} G \leq Aut(\overline{N})$, and of the known such doubly transitive groups the simple primitive normal subgroup $\overline{N}$ is itself doubly transitive, unless $\overline{N} \cong PSL_2(8)$ with $|\Omega| = 28$ (see for example §5, note 4 in [10]).

We assume G is not this group and put $N_1 := H \cap N$, so that:

Lemma 5.14 *$(M^G)_N \cong (M_{N_1})^N$.*

PROOF. Since N is transitive on Ω by hypothesis, we can write $G = HN$ (though not necessarily $H \cap N = 1$). Then Mackey's Theorem implies

$$(M^G)_N = (M_{H\cap N})^N = (M_{N_1})^N,$$

since G has only one (H, N)-double coset.

Along similar lines we have:

Lemma 5.15 *Suppose M_{N_1} extends to N but M does not extend to G. Then:*

- (i) *if G has a simple heart, then M^G is irreducible; and*
- (ii) *if the heart of G over K has no 1-dimensional composition factors, neither does M^G.*

PROOF. If M_{N_1} extends to some KN-module U, we have

$$(M^G)_N \cong (M_{N_1})^N \cong U \otimes K\Omega,$$

by 5.3. Since G has a simple heart, $K\Omega$ has at most three composition factors, by 4.2, at least one of which the trivial module. Now M^G contains a 1-dimensional submodule V if and only if $V_H \cong M$, that is, M extends to G. But M does not extend to G, therefore M^G has no 1-dimensional submodule or factor module and M^G must be irreducible, proving (i). The second part now easily follows, since a 1-dimensional composition factor of M^G is one for $(M^G)_N$.

Theorem 5.16 *We either have:*

- (i) *$(M_{N_1})^N$ is irreducible, so M^G is irreducible;*
- (ii) *$(M_{N_1})^N$ is reducible, so M^G is reducible (but neither is completely reducible);*
- (iii) *$(M_{N_1})^N = U_1 \oplus U_2$ is completely reducible, where $U_1^x = U_2$ for some $x \in G$, so M^G is irreducible; or*
- (iv) *$(M_{N_1})^N = U_1 \oplus U_2$ is completely reducible, where $U_1^x \neq U_2$ for all $x \in G$, so M^G is reducible.*

PROOF. (i) By 5.14, $(M^G)_N \cong (M_{N_1})^N$ so it is clear M^G is irreducible whenever $(M_{N_1})^N$ is.

(ii) If M^G is irreducible then $(M^G)_N$ is completely reducible, expressible as a direct sum of conjugate simple KN-modules, by Clifford's Theorem. So M^G has to be reducible.

(iii) Suppose $V \leq M^G$ is a non-zero irreducible submodule, so that V_N is completely reducible by Clifford's Theorem. If $U_1 \leq V_N$ then $U_1^x \leq V_N^x = V_N$, since V is a G-module. But then $V_N = U_1 \oplus U_2$ and $V = M^G$ is irreducible.

(iv) Let $V_1 \leq M^G$ be a non-zero irreducible submodule as in the proof of *(iii)*, and $(V_1)_N = U_1$ (say). Then $U_2 \not\leq (V_1)_N$ while U_1 extends to $V_1 \leq M^G$.

Note that, since none of the material above utilises the faithfulness of the action of G, 5.16 holds for extensions of doubly transitive groups.

Example 5.17 *Suppose* $\overline{N} = PSL_2(q)$ *with* $q \equiv 1 \pmod 4$, *and* $N_1 = H \cap N$ *with* H *as usual. Then if* $p \neq 2$ *there exists a unique non-trivial 1-dimensional* KN_1*-module* M_{N_1} *affording* λ, *with* $\lambda(n) = \pm 1$ *for all* $n \in N_1$. *If* $p \nmid |N|$ *then* $(M_{N_1})^N \cong U_1 \oplus U_2$, *where* $dim(U_i) = \frac{1}{2}(q+1)$ $(i = 1, 2)$ *and* $U_1^x \cong U_2$ *for some* $x \in Aut(N) \geq G$. *In other words, if* $U_1^x \cong U_2$ *for some* $x \in G$, *Clifford's Theorem implies that* M^G *is irreducible.*

By Tables 2.4 and 2.3, the only situation in which 5.16*(iii)* might apply appears in the following corollary, which applies only to those doubly transitive G such that $\operatorname{Ker} G \leq Z(G)$.

Corollary 5.18 *Suppose* N *is a simple group or a finite central extension thereof. Then* M^G *is irreducible if and only if* $(M_{N_1})^N$ *is irreducible, except in the case where* M *and* $\overline{N}$ *are as in Example 5.17.*

Because of Corollary 5.18 we concentrate on those doubly transitive G that are simple, or perfect central extensions thereof.

The *dual module* of a KG-module V is the module with underlying vector space $V^* := Hom_K(V, K)$ of linear functionals, under point-wise addition and scalar multiplication, with G acting on $\psi \in V^*$ by $(\psi x)(v) = \psi(vx^{-1})$ for all $x \in G$, $v \in V$, extended linearly to KG. In particular, is easy to see that if M is a 1-dimensional KG-module affording λ, then the dual module M^* affords λ^{-1}, where $\lambda^{-1}(x) = \lambda(x^{-1})$ of course.

Let M be a KH-module, for some subgroup $H \leq G$.

Theorem 5.19 (43.9, [15]) *We have* $(M^G)^* \cong (M^*)^G$ *as* KG*-modules. In particular,* M^G *is self-dual whenever* M *is self-dual.*

Corollary 5.20 *With* G *acting transitively on* Ω, *the modules* $K\Omega$ *and* $\mathcal{D}$ *are* *self-dual.*

Lemma 5.21 *Suppose G acts 2-transitively on Ω, with H the stabiliser of a point in this action. Let M be a self-dual KH-module. Then either:*

- (i) $M_D \ncong M_D^t$ *and* M^G *is irreducible;*
- (ii) $M_D \ncong M_D^t$ *and* $soc(M^G) = U_1 \oplus \ldots \oplus U_s$*, with* $U_i^* \ncong U_j$ *for all* j*;*
- (iii) $M_D \cong M_D^t$ *and* $M^G = U_1 \oplus U_2$*, where* U_i *is indecomposable* $(i = 1, 2)$ *and* $i_{KG}(U_1, U_2) = 0$*; or*
- (iv) $M_D \cong M_D^t$ *and for some non-zero submodule* $U < M^G$*,* U *is self-dual and appears as a top and bottom composition factor of* M^G*.*

PROOF. Suppose $M_D \ncong M_D^t$ and M^G is reducible. Then M^G is self-dual by 5.19, and the irreducible summands of $soc(M^G)^*$ appear as top composition factors of M^G. However, $i_{KG}(M^G, M^G) = 1$ by 5.8, so that none of these factors can be isomorphic to a summand of $soc(M^G)$.

Now suppose $M_D \cong M_D^t$. Then $i_{KG}(M^G, M^G) = 2$ by 5.8, so if $M^G = U_1 \oplus U_2$ with U_i indecomposable for $i = 1, 2$, we must have $i_{KG}(U_1, U_2) = 0$. So assume M^G is indecomposable. Then there exists some submodule $V \leq M^G$ such that $M^G/V \cong U \leq M^G$. Since M^G is self-dual (as above), we also have that $U^* \leq M^G$. Should $dim(U + U^*) = dim(M^G)$, then $M^G = U \oplus U^*$, contrary to assumption. Now suppose $U \neq U^*$, so that there there exist 3 endomorphisms $\pi_0 = 1, \pi_1, \pi_2$ of M^G mapping onto M^G, U, U^* (respectively). But then $i_{KG}(M^G, M^G) > 2$ (contradicting 5.8), since $dim(U+U^*) \neq dim(M^G)$ implies π_2 is not a linear combination of π_0, π_1. Therefore $U = U^*$ is self-dual.

Note that, in part (iii) above, we must also have $i_{KG}(U_2, U_1) = 0$. But by assumption, either $U_1^* \cong U_2$ or U_1, U_2 are self-dual, so this equivalent to $i_{KG}(U_1, U_2) = 0$.

6 Groups of Suzuki Type

The following definition can be found in XII 2.1 [21].

Definition 6.1 *A finite permutation group G is said to be of Suzuki type if:*

- (i) *G acts (not necessarily faithfully) doubly transitively on a set Ω, where* $|\Omega| > 3$*;*
- (ii) *G has no regular normal subgroup;*
- (iii) *the stabiliser $H = G_1$ of some point $1 \in \Omega$ has a normal subgroup L acting regularly on* $\Omega \setminus \{1\}$*;*

(*iv*) $|L| = q^f$ *for some prime power* q^f; *and*
(*v*) L *is a characteristic subgroup of* H.

Many doubly transitive groups are of Suzuki type, for example, all the groups appearing in Table 6.10 are such.

G	Comment
$SL_2(q), PSL_2(q)$	$q \geq 4$
$Sz(q)$	$q \geq 8$
$R(q)$	$q > 3$
$SU_3(q), PSU_3(q)$	$q > 2$

Table 6.10
Examples of groups of Suzuki type.

An example of the representation theory of such groups is:

Lemma 6.2 (8(a, b), [28]) *Let* G *be doubly transitive on* Ω, *with* $|\Omega| = n$, *and* F *a field of characteristic* $p \neq 0$ *such that* $p \nmid n$. *Then the heart over* F *of* G *on* Ω *is simple if:*

(*i*) $n = p^f + 1$, *for some* $f \geq 1$; *or*
(*ii*) *the stabiliser* G_1 *of* $1 \in \Omega$ *contains a subgroup* L *transitive on* $\Omega \setminus \{1\}$, *for which* $p \nmid |L|/(n-1)$.

In particular, by 4.5 we have $soc(K\Omega) = \mathcal{C}$ whenever $p \mid n$. In view of Problem 1.9 therefore, the only factor modules of $K\Omega$ for a group of Suzuki type are $\mathcal{C}, \mathcal{C}^{\perp}$ if $p \nmid n$, and $\mathcal{C}$ otherwise.

In addition to the notation of Hypothesis 5.1 and the definition of a Suzuki group, we will assume:

Hypothesis 6.3 *Assume that:*

(*i*) $H = DL$, *where* $L \trianglelefteq H$ *is the subgroup of* H *acting regularly on* $\Omega \setminus \{1\}$;
(*ii*) D *induces at least one fixed-point-free automorphism of* L;
(*iii*) L *is a Sylow subgroup of* G *with* $|L| = q^f$ *for some prime power* q^f;
(*iv*) D *is cyclic; and*
(*v*) *with* $d = |D|$, *we have* $(d, q) = 1$.

Assumptions (*iii*) through (*v*) are not as restrictive as they may appear: all the groups in Table 6.10 satisfy these conditions.

Define the subgroup

$$T := \{d \in D : d^t = d\} \trianglelefteq D$$

to be the set of elements fixed by the outermorphism of D induced by t, and let $S \leq D$ be such that $D/S \cong T$. Let us write $r = |T|, s = |S|$. We have (20.3 [19]):

Lemma 6.4 *There are r distinct reducible characters λ^G, where λ is a linear character of H lifted from H/SL. The remaining $(s-1)r$ characters λ of H induce to $\frac{1}{2}(s-1)r$ distinct irreducible characters λ^G of G.*

Recall that t centralises T, and note that $\operatorname{Ker} G \leq T \leq D$ because of this. The following technical assumption is made in addition to the above.

Hypothesis 6.5 *The largest subgroup $X \leq D$ for which we have $C_L(X) \neq 1$ is precisely T, writing $L_1 := C_L(T)$.*

Then it can be deduced that:

Theorem 6.6 (29.9, [19]) *Suppose $M \cong M^t$ as KD-modules. Then M^G is indecomposable if and only if $p \mid |\Omega|$.*

Theorem 6.7 (33.7, [19]) *Suppose $M \ncong M^t$ as KD-modules. Then M^G is irreducible if and only if $p \nmid q$.*

Theorem 6.8 (33.9, [19]) *Suppose $M \cong M^t$ as FD-modules with $p \mid |D|$. Then if $p \nmid |\Omega|$, and $p \mid |S|$ or $C_L(T) = 1$, we have $M^G = U_1 \oplus U_2$, with U_1, U_2 irreducible.*

7 Calculations

We will assume (K, R, k) to be a *p-modular system*, that is, a discrete valuation ring R with field of quotients K, maximal ideal $\mathcal{P} = \pi R$ and residue class field $k = R/\mathcal{P}$ of characteristic p. We choose K, k sufficiently large for them to be splitting fields for G of characteristics $0, p$ respectively. Let F denote an arbitrary field of characteristic p. We recall that:

Theorem 7.1 (9.14, [22]) *Let U be an absolutely irreducible EG-module, where E has characteristic $p \neq 0$. Suppose that U affords the character χ and that $\chi(x) \in F$ for all $x \in G$, where F is some subfield of E. Then U is similar to $V^E = U \otimes_F E$ for some absolutely irreducible FG-module V.*

Lemma 7.2 *If λ is a linear representation of H over R, then λ^π is a linear representation of H over k. Furthermore, every linear representation of H over k arises in this way and the representations $(\lambda^G)^\pi, (\lambda^\pi)^G$ of G over k are the same.*

PROOF. Since $\operatorname{Im}\lambda$ consists of mth roots of unity in R, where $m = |H/H'|$, $\operatorname{Im}\lambda^{\pi}$ consists of m'th roots of unity in k, where $m = p^a m'$ and $p \nmid m'$. Clearly every linear k-representation of H arises this way.

Now let $\{f \otimes g_1, \dots f \otimes g_n\}$ be a basis for M^G, where M affords λ, the g_i form a right transversal for H in G, and $M = Rf$ for some basis vector f. With respect to this basis, the matrix representation λ^G afforded by M^G has $(\lambda^G(x))_{ij} = \dot{\lambda}(g_i x g_j^{-1})$ for all i, j and $x \in G$. Here $\dot{\lambda}(x) = \lambda(x)$ if $x \in H$ and 0 otherwise. The last result now easily follows by applying π to the matrix entries.

We will adopt notation based on that used in the Atlas: $22a + 154b$ denotes the sum of the first ('a') character of degree 22 and the second ('b') character of degree 154, as they appear in order (from top to bottom) in [13]. Also, $1a+10a+10b+1a$ is contracted to $1aa+10ab$, although we sometimes forgo this convention to highlight the order of factors of a possible composition series. Lastly, $22a + 154b, 154c$ denotes a list of characters $22a + 154b$, $22a + 154c$.

Lemma 7.3 *Take $F = K$, and let λ^G be the character induced from the linear character $\lambda \neq 1_H$ of $H \leq G$. Then:*

- *(i)* $\lambda^G(1) = n$ *the degree of* G*;*
- *(ii)* *if* H *is simple,* $\lambda = 1$*;*
- *(iii)* $(1_H)^G = 1_G + \xi$*, both of which take on integer values; and*
- *(iv)* *if* $\lambda^G = \chi_1 + \chi_2$*, then* $\chi_1 \neq \xi \neq \chi_2$ *and* $(\chi_1 + \chi_2)(h) = \lambda(h) \cdot (1_H)^G$ *for all* $h \in H$*.*

These properties are often sufficient to determine the existence (reducibilities) of non-trivial linear characters (induced linear characters) of the point stabiliser for the 2-transitive groups under consideration. We usually omit a proof if the results follow by inspection of the Atlas [13] or the modular atlas [24], or by 5.8, 5.21 or 7.3. Note that if λ^G is irreducible as a K-character, $i_{FG}(M^G, M^G) = 1$ for any field F.

7.1 The alternating groups, A_m

The alternating groups $G = A_m$ are (m-2)-transitive as the totality of even permutations of a set Ω of m elements. The point stabiliser $H = A_{m-1}$ is simple for $m \geq 6$, in which case it is perfect and affords no non-trivial linear representations. Since the group $A_4 \cong AGL_1(4)$ contains a regular normal subgroup we are left to consider $m = 5$.

Note that $A_5 \cong PSL_2(4)$ acts on the the projective line $PG_1(4)$ and is a group of Suzuki type in this action. Therefore M^G is reducible precisely when $p = 2$, by 6.7, while H affords a non-trivial M when $F \geq GF(4)$ since it is easy to see $H/H' \cong GF(3)$. By inspection of the modular atlas [24], M^G affords the Brauer character $2a + 1a + 2b$ when $p = 2$, where $2a$ is the dual of $2b$ and may be realised over $GF(4)$, by 7.1. The trivial module cannot appear in the socle of M^G since $i_{FG}(F, M^G) = i_{FG}(F_H, M) = 0$ by Frobenius Reciprocity, nor is it a top composition factor by a similar argument. Since $i_{FG}(M^G, M^G) = 1$, M^G has a unique composition series, and the duality of $M^G, (M^*)^G$ follows from 5.19.

By examining the presentation of $2.S_m$ (see 2.12.3 [27]), it is easy to see $2.S_{m-1}$ is contained in $2.S_m$, so:

Lemma 7.4 *For $m \geq 5$, the point stabilisers of $2.A_m$ acting on Ω are isomorphic to $2.A_{m-1}$, which is perfect.*

This leaves $m = 6, 7$, for which A_m has the covering group $6.A_m$.

Lemma 7.5 *A point stabiliser of $3.A_6$ is congruent to $\mathbb{Z}_3 \times A_5$, while for $6.A_6$ it is congruent to $\mathbb{Z}_3 \times 2.A_5$.*

PROOF. Since A_5 has the Schur multiplier $\mathbb{Z}_2$, the point stabiliser of $3.A_6$ must be isomorphic to $3 \times A_5$. However, the point stabiliser of $2.A_6$ is isomorphic to $2.A_5$ by 7.4, and so $6.A_6$ contains a point stabiliser isomorphic to $3 \times 2.A_5$.

Lemma 7.6 *The point stabiliser of $3.A_6$ affords two non-trivial linear characters λ, λ^{-1} which induce to distinct and irreducible characters of G.*

PROOF. By inspection of the ATLAS [13], $3a + 3b$ is a possible λ^G decomposition. However, $\lambda^G(x) = 0$ if x has order 4 (because $2.A_5$ contains no elements of order 4), while $3a + 3b$ is non-zero on this class. Therefore, upon expansion of the character table (to include the second cohort of $3.A_6$), $6a, \overline{6a}$ are the only possibilities for λ^G.

Lemma 7.7 *A point stabiliser H of $3.A_6$ affords two non-trivial linear FH-modules M, M^*, and both are reducible upon induction to G precisely when $F \geq GF(4)$. In this case M^G contains a unique irreducible submodule U with $dim(U) = 3$, and $M^G/U \cong U^*$. Moreover, $(M^*)^G$ is the dual of M^G.*

We first consider $G = PSL_m(q)$ with $m \geq 3$ acting on $\Omega = PG_{m-1}(q)$, where $q = p^a$ for some $a \geq 1$.

Lemma 7.8 *Let $\Omega = PG_{m-1}(q)$ be a projective space, $K\Omega$ the corresponding vector space over K of characteristic $p \mid q$. Let Δ be a k-dimensional projective subspace of Ω and $\{\Gamma_1, \ldots, \Gamma_r\}$ a complete set of l-dimensional projective subspaces of Δ, for $l < k$. Then $\sum_{j=1}^{r} \chi_{\Gamma_j} = \chi_\Delta$.*

PROOF. Let $N_e(d,q)$ denote the number of e-dimensional subspaces contained in a d-dimensional space. Then

$$N_e(d,q) = \Pi_{i=1}^{e} \frac{q^{d-i+1} - 1}{q^i - 1},$$

and $N_e(d,q) \equiv 1 \pmod p$. In particular, $r \equiv 1 \pmod p$. It is easy to see that the number of $(l+1)$-dimensional subspaces containing a given 1-dimensional space in a $(k+1)$-dimensional space is equal to $N_l(k,q)$, so that χ_α appears with coefficient 1 in the sum of all χ_{Γ_j} such that $\alpha \in \Gamma_j$, given some $\alpha \in \Delta$. Similarly, the number of $(l+1)$-dimensional subspaces containing a 2-dimensional space in a $(k+1)$-dimensional space is $N_{l-1}(k-1,q)$, so such a sum also contains χ_β with coefficient 1. The number of Γ_j's containing β that do not contain α is then $0 \pmod p$.

Lemma 7.9 *Let V_i be the subspace of $F\Omega$ where V_i is generated by the characteristic functions of all i-dimensional subspaces of Ω, for $i = 1, \ldots, m-1$. Then:*

- (*i*) $F\Omega = V_0 \geq V_1 \geq \ldots \geq V_{m-1} = \mathcal{C}$;
- (*ii*) *for all* $i = 0, 1, \ldots, m-1$, $V_i = \mathcal{C} \oplus W_i$ *where W_i is generated by the characteristic functions of all complements of i-dimensional subspaces; and*
- (*iii*) V_{m-2} *has dimension* $\binom{p+m-2}{m-1}^a + 1$, *and in particular,* V_{m-2} *is a proper subspace of* $F\Omega$.

PROOF. (*i*) Clearly $V_0 = F\Omega$ and $V_{m-1} = \mathcal{C}$. Let Δ be an $(i+1)$-dimensional projective subspace of Ω. We must show that $\chi_\Delta \in V_{i+1}$ is in V_i. But this follows from 7.8, claiming $\sum_\Gamma \chi_\Gamma = \chi_\Delta$, where the sum is taken over all i-dimensional projective subspaces contained in Δ.

(*ii*) If $i = m-1$ then $W_{m-1} = 0$ and $V_{m-1} = \mathcal{C}$ and we are done. Suppose $V_j = \mathcal{C} \oplus W_j$ is true for $i < j < m-1$. We have $\mathcal{C} = W_{m-1} \leq V_i$ by (*i*), so the complement W_i is in V_i too. Therefore $\mathcal{C} + W_i \leq V_i$ and so $\mathcal{C} + W_i = V_i$. We have to show this is a direct sum, that is, $\mathcal{C} \cap W_i = 0$. This is equivalent to $W_i \neq V_i$, or $W_i^{\perp} \neq V_i^{\perp}$. Since $p \mid |\overline{\Delta}|$, where $\overline{\Delta}$ is the complement of an i-dimensional projective subspace, but $p \nmid |\Delta|$, we have $\mathcal{C} \leq W_i^{\perp}$ but $\mathcal{C} \not\leq V_i^{\perp}$.

(iii) This is proved in [36].

The following is a generalisation of 5.1 [23].

Theorem 7.10 (37.4, [19]) *Let $G = PSL_m(q)$ act on $\Omega = PG_{m-1}(q)$, for $m \geq 3$. Let $F\Omega$ be the corresponding permutation module over F of characterstic $p \mid q$. Suppose $U \leq F\Omega$ is a non-zero submodule, $U \neq \mathcal{C}$. Then $W \leq U$, where $W = W_{m-2}$ is generated by the characteristic functions of the complements of hyperplanes in Ω, and*

$$W + \mathcal{C} = \langle \chi_\Delta : \Delta \leq \Omega, \textit{ where } \Delta \textit{ is } (m-2)\textit{-dimensional} \rangle$$

is the design module of G.

We now turn to the non-trivial FH-modules M, and consider first $G = SL_2(q)$, with $q \neq 2, 3$ so that $SL_2(q)$ is perfect (see 4.5 [40]). Clearly $SL_2(q)$ contains the point stabiliser $H = G_{[0,1]} = DL$ of order $(q-1)q$, where D consists of all diagonal matrices of determinant 1, and L of all upper unitriangular matrices. From $q \neq 2, 3$ it follows D induces a fixed-point-free automorphism of L, so that $H' \cong L$, $H/H' \cong D$ and H affords at most $q - 1$ linear representations.

Lemma 7.11 *The groups $G = SL_2(q)$ are of Suzuki type.*

PROOF. We have G doubly transitive on Ω, where $|\Omega| = q + 1 > 4$. Also, G has no regular normal subgroup, since $G/Z(G)$ is simple, while the stabiliser H of $[0, 1] \in \Omega$ contains the regular normal subgroup $L \cong GF(q)$, since $[1, 0]l_\beta = [1, \beta]$ for all $\beta \in GF(q)$. This L is a characteristic subgroup of H since $(|D|, |L|) = 1$.

We take the element t to be such that $[0, 1]^t = [1, 0]$. Then $C_t(D) = T = \langle d_{-1} \rangle$ if q is odd, and $C_t(D) = T = 1$ if q is even. In particular, if q is odd H affords a unique non-trivial linear representation λ into F (provided $char(F) \neq 2$) such that $\lambda^t = \lambda$ on D. Note that $|S| = \frac{1}{2}(q-1)$.

We suppose M is a FH-module affording λ, and consider p dividing each factor of $|G| = (q+1)(q-1)q$ in turn. If $p \mid q - 1$ then $p \mid |S|$ and $p \nmid |\Omega| = q + 1$, so that we may apply 6.8 to deduce:

Lemma 7.12 *If $p \neq 2$ and $p \mid q-1$, then $M^G = U_1 \oplus U_2$, with U_1, U_2 irreducible. Here $dim(U_1) = dim(U_2) = \frac{1}{2}(q+1)$.*

PROOF. By the above, the first part follows from 6.8, while the dimensions of the summands can be found, for example, in [37].

Next we consider $p \mid q+1$ and take $F = k$ in the above. Then M affords a linear k-representation χ, and there exists some linear R-representation μ of H such that $\mu^\pi = \lambda$, by 7.2. Now μ^G is the character χ^G put forward above, and $\chi^G = \chi_1 + \chi_2$ with $\chi_1(1) = \chi_2(1) = \frac{1}{2}(q+1)$. We have $p \nmid |H|$, because $p \nmid (q-1)q$ and $p \neq 2$, and so χ_1, χ_2 are contained in a block of defect 0:

Lemma 7.13 *If $p \mid q+1$ then $M^G = U_1 \oplus U_2$, with U_1, U_2 irreducible. Here $dim(U_1) = dim(U_2) = \frac{1}{2}(q+1)$.*

For the case $p \mid q$ we refer the reader to [37].

Lemma 7.14 *Let $G = PSL_m(q)$ be a projective special linear group with exceptional Schur multiplier, as listed in Table 3.8. Suppose $\overline{G}$ is a covering group of G, with $\overline{G} \not\cong SL_m(q)$, and $\overline{H}$ the pre-image of the point stabiliser $H \leq G$ in its natural action on Ω. Then all linear characters of $\overline{H}$ are lifted from H.*

PROOF. This follows from direct calculations using Gap [20], and in some cases by inspection of the Atlas [13]. For example, $3.PSL_2(9)$ affords no characters (or sum of two) of degree 10 that are not lifted from $PSL_2(9)$.

Next we consider $m \geq 3$. Let $\alpha = [0, \ldots, 0, 1]$ be the projective point with stabiliser $H \leq G = SL_m(q)$. Denote by $k(A, v, \gamma)$ the matrix

$$k(A, v, \gamma) := \begin{pmatrix} A & v \\ 0 & \gamma \end{pmatrix} \in GL_m(q), \tag{7.1}$$

where $A \in GL_{m-1}(q)$, $v \in GF(q)^{m-1}$ and $\gamma \in GF(q)^*$. Then the group $H \leq SL_m(q)$ consists of the matrices of the form $k(A, v, \gamma)$ in (7.1), with $\gamma = det(A)^{-1}$. We will abbreviate those $k(A, v, \gamma)$ of determinant 1 to $k(A, v)$.

Lemma 7.15 *For any $A, B \in GL_{m-1}(q)$ and $v, w \in GF(q)^{m-1}$, we have:*

(i) $k(A, v)k(B, w) = k(AB, Aw + det(B)^{-1}v)$;
(ii) $L = \{k(I, v) : v \in GF(q)^{m-1}\} \trianglelefteq H$; *and*

(iii) *H' is generated by matrices of the form $k([A,B],u)$, where*

$$[A,B] := ABA^{-1}B^{-1}$$

and $u \in GF(q)^{m-1}$ is arbitrary.

It follows that $H/H' \cong GF(q)^*$, except when $(m,q) = (3,2)$, in which case $H/H' \cong GF(3)^*$. Next put $\beta = [0,\dots,0,1,0]$ and let $t \in SL_m(q)$ be the element

$$\begin{pmatrix} I_{m-2} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

of order 4, such that $\alpha^t = \beta$ and $\beta^t = \alpha$.

With $D := H \cap t^{-1}Ht$ we abuse notation somewhat by treating the $k(B,v,\gamma_2)$ as matrices in $GL_{m-1}(q)$, to note that $D \leq SL_m(q)$ consists of the matrices of the form

$$l(k(B,v,\gamma_2),w) := \begin{pmatrix} B & v & w \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_1 \end{pmatrix}, \tag{7.2}$$

with $B \in GL_{m-2}(q)$, $v,w \in GF(q)^{m-2}$, $\gamma_1,\gamma_2 \in GF(q)^*$ and $\gamma_1\gamma_2 = det(B)^{-1}$.

Lemma 7.16 *When $(m,q) \neq (4,2)$, then D' consists of the matrices of the form $l(k(B,v,1),w)$, for all $B \in SL_{m-2}(q)$ and $v,w \in GF(q)^{m-2}$.*

By 7.16, we have $D/D' \cong GF(q)^* \times GF(q)^*$ whenever $(m,q) \neq (4,2)$. In this case, the linear representations of D are fully determined by the values of γ_1,γ_2 of the matrix $l(k(B,v,\gamma_2),w) \in D$. With

$$\begin{aligned} L_{\gamma_1} &= \{l(k(B,v,\gamma_2),w) \in D : det(B)\gamma_2 = 1\} \\ L_{\gamma_2} &= \{l(k(B,v,1),w) \in D : det(B)\gamma_1 = 1\}, \end{aligned}$$

write $\mu_{\alpha\beta}$ for an arbitrary linear representation of D, where $\mu_{\alpha\beta} := \alpha \cdot \beta$ for linear representations α,β lifted from $D/L_{\gamma_1}, D/L_{\gamma_2}$ respectively. Note that, prior to lifting, $\alpha,\beta \in irr(GF(q)^*)$.

Theorem 7.17 *Let $G = SL_m(q)$ with $m \geq 3$, and λ some linear representation of H. Then $\lambda \neq \lambda^t$ on D if and only if $\lambda \neq 1_H$.*

PROOF. We may assume $q \neq 2$, so that all linear representations of D are of the form $\mu_{\alpha\beta}$ above. Let $\lambda \neq 1_H$ be a linear representation of H. Then

$\lambda_D = \mu_{\alpha 1_D}$ for some α as above, since λ_D is fully determined by the value of $det(B)\gamma_2 = \gamma_1^{-1}$. With t as above,

$$l(k(B, v, \gamma_2), w)^t = l(k(B, w, \gamma_1), -v), \tag{7.3}$$

for some B, v, γ_2, w and γ_1 as in (7.2). Let l be the element of (7.3), so that

$$\lambda_D^t(l) = \mu_{\alpha 1_D}^t(l) = \alpha(\gamma_2) = \mu_{1_D \alpha}(l).$$

Therefore, $\mu^t = \mu$ on D if and only if $\alpha = 1_D$, in which case $\lambda = 1_H$.

Put $R = \{k(A, 0) : A \in GL_{m-1}(q)\}$.

Lemma 7.18 *When $(m, q) \neq (3, 2)$, the map $\mu_H \mapsto \mu_R$ is a bijection between the sets of linear representations of H and R.*

PROOF. By 7.15(ii), H', R' contain the matrices of the form $k(A, v, 1)$, $k(A, 0, 1)$ respectively, for all $A \in SL_{m-1}(q)$, $v \in GF(q)^{m-1}$, unless $(m, q) = (3, 2)$. Then $H/H' \cong R/R' \cong GF(q)^*$, $R' = H' \cap R$ and the result follows by Proposition 5.2.

It is easy to see R has the orbits $[0, 1]$, $\Delta_0 = [x, 0]^R$ and $\Delta_1 = [x, 1]^R$ on $\Omega = PG_{m-1}(q)$. Moreover, the actions of R on Δ_0, Δ_1 are permutation isomorphic to $GL_{m-1}(q), \bar{R}$ on $PG_{m-2}(q), V_{m-1}(q) \setminus \{0\}$ respectively, where

$$\bar{R} = \{A \in GL_{m-1}(q) : det(A) \in (GF(q)^*)^2\}. \tag{7.4}$$

Therefore H has two (D, R)-double cosets, and we will use

$$x = \begin{pmatrix} I & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7.5}$$

as the double coset representative in $H \setminus D \cup R$. Then $R_0 = D \cap R$ is a point stabiliser for R on Δ_0, while $R_1 = D^x \cap R$ is one for R on Δ_1.

Lemma 7.19 *When $(m, q) \neq (3, 2), (4, 2)$, the group $R_1 = D^x \cap R$ has index $q - 1$ in R, and the map $\mu_R \mapsto \mu_{R_1}$ is a bijection between the sets of linear representations of R and R_1.*

PROOF. As in the proof of 7.18, R' contains the matrices of the form $k(A,0,1)$, for all $A \in SL_{m-1}(q)$, unless $(m,q) = (3,2)$. With x as in (7.5), the group $R_1 = D^x \cap R$ consists of the matrices of the form

$$l(k(B,u,\gamma_1),0) = \begin{pmatrix} B & u & 0 \\ 0 & \gamma_1 & 0 \\ 0 & 0 & \gamma_1 \end{pmatrix}, \tag{7.6}$$

for all $B \in GL_{m-2}(q)$, $u \in GF(q)^{m-2}$ and $\gamma_1 \in GF(q)^*$, such that $det(B)^{-1} = \gamma_1^2$. Fixing $\gamma_1 \in GF(q)^*$, there are $|SL_{m-2}(q)|$ choices for B, so that R_1 has index $q-1$ in R. Analogous to 7.15(iii), ${R_1}'$ consists of the matrices of the form $l(k(B,u,1),0)$, for all $B \in SL_{m-2}(q)$ and $u \in GF(q)^{m-2}$, except when $(m-2,q) = (2,2)$. Thus

$$R_1/{R_1}' \cong R/R' \cong GF(q)^*,$$

and ${R_1}' = R' \cap R_1 = R' \cap D^x$. The result now follows by Proposition 5.2.

Note that R_0 consists of those $l(k(B,v,\gamma_2),w) \in D$ such that $w = 0$, so that by 7.15(ii), it is easy to see $R_1 \trianglelefteq R_0$. We will write $F\Delta_1 = (F_{R_1})^R$, for the permutation module of R acting on $\Delta_1 = R/R_1$. This action is permutation isomorphic to $\bar{R}$ on $V_{m-1}(q) \setminus \{0\}$, with $\bar{R}$ as in (7.4).

Lemma 7.20 *When $(m,q) \neq (3,2)$, we have $F\Delta_1 \cong \oplus_i (M_i)^R$, for the $q-1$ 1-dimensional FR_0-modules M_i, lifted from R_0/R_1. Furthermore, the non-trivial M_i do not extend to R.*

PROOF. Recall that $R_1 \leq R$ comprises the elements $l(k(B,u,\gamma_1),0)$ in (7.6). It is easy to see $R_0/R_1 \cong GF(q)^*$; fixing $\gamma_1, \gamma_2 \in GF(q)^*$, there exist $|SL_{m-2}(q)|$ choices for B in $l(k(B,u,\gamma_2),0)$, while forcing $\gamma_1 = \gamma_2$ gives rise to R_1. Then $(F_{R_1})^{R_0} = \oplus_i M_i$, where M_i are the $q-1$ linear representations of R_0 lifted from $R_0/R_1 \cong GF(q)^*$, using Frobenius Reciprocity. By transitivity of induction,

$$F\Delta_1 \cong ((F_{R_1})^{R_0})^R \cong \oplus_i (M_i)^R.$$

Finally, the non-trivial M_i do not extend to R since the 1-dimensional modules of R are in bijective correspondence with those of R_1 by 7.19, while $(M_i)_{R_1} \cong F_{R_1}$ for all non-trivial M_i.

Lemma 7.21 *Suppose $(m,q) \neq (3,2),(3,3)$, and let M be a 1-dimensional FH-module such that $M_D \not\cong M_D^t$. Then the FG-module M^G is irreducible whenever $p \nmid q$.*

PROOF. If $(m, q) \neq (3, 2)$ and $M_D \ncong M_D^t$ we must have $q > 2$. By 5.13, it suffices to show $(M_D^t)^H$ does not contain M as a composition factor. However, with R as above, 7.18 implies we may equally consider the composition factors of the module $((M_D^t)^H)_R$ instead. Now, H has two (D, R)-double cosets, so Mackey's Theorem provides

$$((M_D^t)^H)_R \cong (M^t_{D\cap R})^R \oplus (M^{tx}_{D^x \cap R})^R, \tag{7.7}$$

with $x \in H \setminus D \cup R$ as in (7.5). Applying Frobenius Reciprocity then gives

$$\begin{aligned} 0 &= i_{FR}(M_R, ((M_D^t)^H)_R) \\ &= i_{FR}(M_R, (M^t_{D\cap R})^R) + i_{FR}(M_R, (M^{tx}_{D^x\cap R})^R) \\ &= i_{FR_0}(M, M^t) + i_{FR_1}(M, M^{tx}), \end{aligned} \tag{7.8}$$

by 7.18 and the fact that $M \ncong M^t$ as FD-modules.

Recall $R \cong GL_{m-1}(q)$ acting on $PG_{m-2}(q)$, with R_0 its point stabiliser. Therefore, the first summand of (7.7) may be written $({M'}_{R_0})^{GL_{m-1}(q)}$, for some 1-dimensional FR_0-module M'. On the other hand, we may apply 7.19 to the second summand of (7.7) to obtain

$$(M^{tx}_{R_1})^R \cong U \otimes (F_{R_1})^R \cong U \otimes F\Delta_1,$$

for some 1-dimensional FR-module U, by 5.3. Here $U \cong M^{tx}$ as FR_1-modules.

Suppose now $m = 3$. We first consider $({M'}_{R_0})^{GL_2(q)}$ as above, and write

$$(({M'}_{R_0})^{GL_2(q)})_{SL_2(q)} \cong ({M'}_{R_0 \cap SL_2(q)})^{SL_2(q)},$$

analogous to 5.14. The heart of $SL_2(q)$ has no non-trivial 1-dimensional composition factors by §3(F) [31], unless $m - 1 = 2$ and $q = 3$. We will assume $(m, q) \neq (3, 3)$. Therefore, if ${M'}_{R_0}$ extends to $GL_2(q)$, $({M'}_{R_0})^{GL_2(q)}$ contains only the 1-dimensional composition factor V, where $V_{R_0} \cong M'$. By (7.8) above, it follows this composition factor is not M_R.

Now suppose ${M'}_{R_0}$ does not extend to $GL_2(q)$. Then 5.15(ii) implies that if ${M'}_{R_0\cap SL_2(q)}$ extends to $SL_2(q)$, then $({M'}_{R_0})^{GL_2(q)}$ contains no non-trivial 1-dimensional composition factors. On the other hand, if ${M'}_{R_0\cap SL_2(q)}$ does not extend to $SL_2(q)$ (and $p \nmid q$), the latter module is irreducible by 7.11, 7.12, 7.13 and 5.16.

Next we examine $U \otimes F\Delta_1$, as above. By 7.20,

$$F\Delta_1 \cong (F_{R_1})^R \cong \oplus_i (M_i)^R,$$

where the $q - 1$ M_i are 1-dimensional and do not extend to R. Since they are FR_0-modules, the module $F\Delta_1$ contains no non-trivial 1-dimensional composition factors, by the discussion above. Therefore neither does $U \otimes F\Delta_1$, by (7.8).

Finally suppose $m > 3$ and the result holds for $m-1$. Then we may treat $X = ({M'}_{R_0})^{GL_{m-1}(q)}$ as above, that is:

(i) $X = F(PG_{m-2}(q))$ has a simple heart by [31]; and

(ii) ${M'}_{R_0}$ extends to $GL_{m-1}(q)$ so that X contains only 1-dimensional composition factors distinct from M_R, by 5.3 and (7.8); or

(iii) ${M'}_{R_0}$ does not extend to $GL_{m-1}(q)$ so that either X is irreducible by induction, or X lacks non-trivial 1-dimensional composition factors, by 5.15(ii) and (7.8).

Similarly, the M_i in

$$(M^{tx}_{R_1})^R \cong \oplus_i (U \otimes (M_i)^R)$$

are 1-dimensional FR_0-modules (and $R \cong GL_{m-1}(q)$) by 7.20, and these summands appear as (i), (ii) or (iii) above. Thus the indecomposable summands of (7.7) contain no composition factors isomorphic to M_R and the proof is complete.

However, by inspection of the Atlas [13], modular atlas [24] or using Gap [20] we find:

Lemma 7.22 *Suppose* $(m,q) = (3,2)$ *or* $(3,3)$*, and* M *is a 1-dimensional* FH*-module such that* $M_D \not\cong M_D^t$*. Then the* FG*-module* M^G *is irreducible if and only if* $p \nmid q$*.*

To prove the converse to 7.21, we will describe a module $V_m^r(q)$ and a homomorphism $\phi_m^r : M^G \to V_m^r(q)$, with $\operatorname{Ker}\phi \neq \{0\}$. Let

$$X_1 = [1,0,\ldots,0], X_2 = [0,1,0,\ldots,0], \ldots, X_m = [0,\ldots,0,1]$$

be a basis for $V_m(q)$. Write $V_m^r(q) = F[X_1,\ldots,X_m]^r$ for the ring of homogeneous polynomials in $X_1,\ldots,X_m$ of degree r over $GF(q)$. Then $V_m^r(q)$ is a $SL_m(q)$-module.

Let $H \leq G$ be the stabiliser of $X_1 \in V_m(q)$. Then $|H/H'| = q-1$ implies there is a faithful H/H'-module M' over $GF(q)$ which, once lifted to H, is isomorphic to the H-module spanned by X_1. Moreover, the space spanned by X_1^r is isomorphic to the r-fold tensor product $\otimes^r M' = M' \otimes \ldots \otimes M'$. With M' spanned by e, the linear transformation

$$\phi_m^r : \otimes^r M' \to V_m^r(q)_H, \quad e \mapsto X_1^r,$$

is a H-module homomorphism, and extends to a (not necessarily onto) G-module homomorphism $\widehat{\phi_m^r} : (\otimes^r M')^G \to V_m^r(q)$ by Frobenius Reciprocity.

Of course any linear H-module over $GF(q)$ is isomorphic to $\otimes^r M'$, for some r, and we will suppose $M \cong \otimes^r M'$ below, with appropriate r in the range $1 \leq r \leq q-1$.

Lemma 7.23 *Suppose* $(m,q) \neq (3,2),(3,3)$*, and let* M *be a 1-dimensional* FH*-module such that* $M_D \not\cong M_D^t$*. Then the* FG*-module* M^G *is reducible whenever* $p \mid q$*.*

PROOF. If we had $m=2$, the module $V_2^r(q)$ would have dimension $r+1$, with $1 \leq r \leq q-1$, while M^G had dimension $q+1$. Therefore $\operatorname{Ker}\widehat{\phi_2^r} \neq 0$, and the result would be true for $m=2$.

Consider now $m \geq 3$. Let $g_\alpha \in SL_m(q)$ be the element

$$g_\alpha : [0,\ldots,0,1] \mapsto [0,\ldots,0,\alpha,1],$$

for all $\alpha \in GF(q)$, and

$$g_\infty : [0,\ldots,0,1] \mapsto [0,\ldots,0,1,0].$$

With M spanned by e as above, consider the images

$$\{\widehat{\phi_m^r}(e \otimes g_\alpha), \widehat{\phi_m^r}(e \otimes g_\infty) : \alpha \in GF(q)\}$$

of the elements of the standard basis under the homomorphism $\widehat{\phi_m^r}$. These map to $(\alpha X_{m-1} + X_m)^r$ and X_{m-1}^r respectively, for all $\alpha \in GF(q)$. By the argument for $m=2$, there is a linear combination of these images equal to $0 \in V_m^r(q)$. Hence $\operatorname{Ker}\widehat{\phi_m^r} \neq \{0\}$, and M^G is reducible.

We summarise the above discussion into the main result of this section, using 7.21, 7.22 and 7.23.

Theorem 7.24 *With* $m \geq 3$*, let* M *be a 1-dimensional* FH*-module such that* $M_D \not\cong M_D^t$*. Then the* FG*-module* M^G *is irreducible if and only if* $p \nmid q$*.*

7.3 Suzuki groups, $Sz(q)$

The Suzuki groups $Sz(q) = {}^2B_2(q)$ were first described by Suzuki in [38], where $q = 2^{2m+1}$ and $m > 0$. We will adopt the notation of XI §3 [21], so that $Sz(q) := \langle s_{a,b}, m_\mu, t\rangle$ for $a,b \in GF(q)$, $\mu \in GF(q)^*$, and $|Sz(q)| = (q^2+1)q^2(q-1)$. For $G = Sz(q)$ it is then easy to see that $H := G_\infty = \langle s_{a,b}, m_\mu\rangle$, $D := G_{\infty 0} = \langle m_\mu \rangle$ and $L := \langle s_{a,b}\rangle$ is a 2-Sylow subgroup of G. Note that $D \cong GF(q)^*$, and L is regular on $\Omega \setminus \{\infty\}$. A straightforward calculation reveals that all three point stabilisers are trivial, so that D induces at least one fixed-point-free automorphism of L.

Lemma 7.25 *The groups $G = Sz(q)$ are of Suzuki type.*

PROOF. We have G doubly transitive on Ω, where $|\Omega| = q^2+1 > 5$. Also, G has no regular normal subgroup, since G is simple, while the point stabiliser $H = G_\infty$ of $\infty \in \Omega$ contains the regular normal subgroup L, and $|L| = 2^f$ for some $f \geq 6$. This L is a characteristic subgroup of H since $(|D|, |L|) = 1$.

The involution t induces the action $\infty \leftrightarrow 0$ and conjugates m_μ to m_μ^{-1}. But $D \cong GF(q)^*$, where q is a power of 2, so $C_t(D) = T = 1$. It follows that H affords no FH-module M such that $M \cong M^t \cong M^*$ as FD-modules.

Finally, $G = Sz(q)$ has a trivial Schur multiplier, unless $q = 8$, in which case it has the Schur multiplier $\mathbb{Z}_2 \times \mathbb{Z}_2$. However:

Lemma 7.26 *Let $G = Sz(8)$ and H be a point stabiliser in the doubly transitive action of G, as before. Let $\overline{H}$ be the pre-image of H in $2^2.Sz(8)$. Then all linear characters of $\overline{H}$ are lifted from H to $\overline{H}$.*

PROOF. By inspection of the character table of $Sz(8)$ (in the ATLAS [13]) there are no characters of $2^2.Sz(8)$ whose degrees might add up to $(1_H)^G(1) = 65$, other than those lifted from $Sz(8)$.

7.4 Ree groups, $R(q)$

The Ree groups $R(q) = {}^2G_2(q)$ were first described by Ree in [33], where $q = 3^{2m+1}$ with $m \geq 0$, and they are simple when $m > 0$. We will adopt the notation of 7.7 [16] (see also XI §13 [21]), so that $R(q) := \langle t, r_{\alpha,\beta,\gamma}, n_\kappa \rangle$ for $\alpha, \beta, \gamma, \kappa \in GF(q)$ with $\kappa \neq 0$, and $|R(q)| = (q^3+1)q^3(q-1)$. For $G = R(q)$ it is then easy to see that $H := G_\infty = \langle r_{\alpha,\beta,\gamma}, n_\kappa \rangle$, $D := G_{\infty 0} = \langle n_\kappa \rangle$ and $L := \langle r_{\alpha,\beta,\gamma} : \alpha, \beta, \gamma \in GF(q) \rangle$ is a 3-Sylow subgroup of G. Note that L acts regulary on $\Omega \setminus \{\infty\}$.

Lemma 7.27 *The groups $G = R(q)$ are of Suzuki type.*

PROOF. We have G doubly transitive on Ω, where $|\Omega| = q^3 + 1 \geq 28$. Also, G has no regular normal subgroup since G is simple for $q > 3$, and $R(3) \cong P\Sigma L_2(8)$ contains the simple group $PSL_2(8)$ with index 3. The point stabiliser $H = G_\infty$ of $\infty \in \Omega$ contains the regular normal subgroup L, and $|L| = 3^f$ for some $f \geq 3$. This L is a characteristic subgroup of H since $(|D|, |L|) = 1$.

Since the involution t induces the action $\infty \leftrightarrow 0$ and conjugates n_κ to n_κ^{-1}, a straightforward calculation reveals that only $n_1, n_{-1} \in D$ centralise an element of the form $r_{0,\beta,0} \in L$. It follows there exists a fixed-point-free automorphism of L whenever $q > 3$, which we will assume. Conversely, the only $X \leq D$ to have $C_L(X) \neq 1$ is T, with $C_L(T) = \langle r_{0,\beta,0} : \beta \in GF(q)\rangle$. So $C_t(D) = T = \langle n_{-1}\rangle$ and $s = |S| = \frac{1}{2}(q-1)$. But $D \cong GF(q)^*$ with q a power of 3, so H affords a unique FH-module M such that $M \cong M^t \cong M^*$ as FD-modules (provided $char(F) \neq 2$).

Table 7.11 summarises the ordinary characters of $R(q)$ and their degrees, and is taken from in [41]. We have denoted $(1_H)^G$ as $1 + \xi$, by λ^G the $\frac{1}{2}(q-3)$ characters induced from non-trivial complex linear characters λ of H, and by $\chi_1 + \chi_2$ the decomposition of the character χ^G, where χ is the unique real linear character of H.

No.	Character	Degree
1	$\xi_1 = 1_G$	1
1	$\xi_2 = \chi_1$	$q^2 - q + 1$
1	$\xi_3 = \xi$	q^3
1	$\xi_4 = \chi_2$	$q(q^2 - q + 1)$
2	ξ_5, ξ_7	$\frac{1}{6}(q-1)r(q+1+r)$
1	ξ_6, ξ_8	$\frac{1}{6}(q-1)r(q+1-r)$
2	ξ_9, ξ_{10}	$\frac{1}{3}r(q^2-1)$
$\frac{1}{2}(q-3)$	$\eta_i, \eta_i' = \lambda^G$	$q^3 + 1$
$\frac{1}{24}(q-3)$	η_j	$(q-1)(q^2-q+1)$
$\frac{1}{8}(q-3)$	η_j'	$(q-1)(q^2-q+1)$
$\frac{1}{6}(q-r)$	η_k^-	$(q^2-1)(q^2+q+r)$
$\frac{1}{6}(q+r)$	η_k^+	$(q^2-1)(q^2+q-r)$

Table 7.11
Characters of $G = R(q)$ and their degrees.

We will assume $p \neq 2$, so that $p \mid |D| = q - 1$ implies $p \mid |S|$. Then $p \nmid n = q^3 + 1$, so that 6.8 applies: $M^G = U_1 \oplus U_2$, with U_1, U_2 irreducible. Here $dim(U_1) = q^2 - q + 1$ and $dim(U_2) = q(q^2 - q + 1)$ by Table 7.11.

It remains to consider $p \mid q^3+1$ and $p \mid q$, though we have not yet tackled the latter. Take $F = k$ in the above, so that M affords a linear k-representation λ, and there exists some linear R-representation μ of H such that $\mu^{\pi} = \lambda$, by 7.2. Now μ^G is the character $\chi = \chi_1 + \chi_2$ with $\chi_1(1) = q^2 - q + 1, \chi_2(1) = q(q^2 - q + 1)$.

By XI 13.2 [21], G has Abelian Hall subgroups H_1, H_2 of orders $q + 1 \pm r$, where $r = 3^{m+1}$ so that $r^2 = 3q$. Thus if $p \mid q^2 - q + 1$ then $p \nmid |H|$ and χ_1, χ_2 are contained in a block of defect 0. In this case:

Lemma 7.28 *If* $p \mid q^2 - q + 1$ *then* $M^G = U_1 \oplus U_2$, *with* U_1, U_2 *irreducible. Here* $dim(U_1) = q^2 - q + 1$ *and* $dim(U_2) = q(q^2 - q + 1)$.

Next we consider $p \mid q+1$. In this case the kG-module M^G is indecomposable by 6.6 and projective since $p > 3$ and $q+1$ is coprime to $(q^2-q+1)(q-1)$. Moreover, the block B containing M^G has a cyclic defect group, as it has a maximal defect group $Q \in Syl_p(G)$ and all p-Sylow subgroups are cyclic for $p > 3$ by XI 13.2 [21]. Using the equivalences of 61.2 [17] (or 56.23, 56.24 in [14]), is possible to establish:

Lemma 7.29 (47.3, [19]) *The block* B *contains the ordinary* K*-characters* χ_1, χ_2 *and* $\frac{1}{2}(p^b - 1)$ *characters of the type* η'_i *in Table 7.11. Here* $\frac{1}{4}(q+1) = p^b m$ *with* $p \nmid m$, *and* $i = jm$ *with* j *in the range* $1 \le j \le \frac{1}{2}(p^b - 1)$.

Lemma 7.30 *If* $p \mid q+1$ *then* M^G *is projective and indecomposable, with* $rad(M^G)/soc(M^G)$ *irreducible. Writing* $\varepsilon = \frac{1}{2}(p^b - 1)$, *the block* B *containing* M^G *affords the decomposition matrix:*

B:	φ_1	φ_2
χ_1	1	0
χ_2	1	1
η'_m	0	1
η'_{2m}	0	1
$\vdots$	$\vdots$	$\vdots$
$\eta'_{\varepsilon m}$	0	1

Here the irreducible Brauer characters φ_1, φ_2 *have degrees* $q^2 - q + 1, (q-1)(q^2 - q + 1)$.

PROOF. By 7.29 the block B, containing the indecomposable kG-module M^G and the K-characters $\mu^G = \chi_1 + \chi_2$, contains a further $\frac{1}{2}(p^b - 1)$ irreducible characters, but only two irreducible Brauer characters. Since M^G is projective by the comments before 7.29, we have $M^G/rad(M^G) \cong soc(M^G)$ (see 6.6 in [1]), and M^G contains contains two copies of the composition factor $soc(M^G)$. Therefore $\chi_1^\pi + \chi_2^\pi = 2\varphi_1 + \ldots$, where $soc(M^G)$ affords φ_1 say, and the conditions on the decomposition numbers in Dade's Theorem (68.1 in [17]) force the described decomposition matrix.

7.5 Unitary groups, $SU_3(q)$

Let $q > 2$ be a prime power V a 3-dimensional vector space over $F = GF(q^2)$. We take $\sigma : a \mapsto a^q$ to be an automorphism of F such that $\sigma^2 = 1$ and $F_0 \cong GF(q)$ is the subfield fixed by σ. We will write a^σ for the image of a under σ.

Now take $\beta : V \times V \to F$ be a σ-hermitian form. Pick a basis $\{e, w, f\}$ for V such that $\beta(e, f) = 1$, $\beta(e, e) = \beta(f, f) = 0$ (i.e. (e, f) is a *hyperbolic pair*), and $\beta(w, w) = 1$. We will work with $SU_3(q)$ and its image $PSU_3(q)$ in terms of matrices over this basis.

The group $G = SU_3(q)$ has a doubly transitive action on the *isotropic points* of $PG_2(r)$ (see for example 10.12 [40]). These are the 1-dimensional spaces $\langle u \rangle$ for $u \in V$, such that $\beta(u, u) = 0$. The group $PSU_3(q)$ is simple for $r = q^2 > 4$ (see 10.15 [40]).

Lemma 7.31 *The point stabiliser $H = G_{\langle f \rangle}$ consists of the products of the elements*

$$h_k = \begin{pmatrix} k & 0 & 0 \\ 0 & \frac{k^\sigma}{k} & 0 \\ 0 & 0 & k^{-\sigma} \end{pmatrix} \text{ and } t_{x,y} = \begin{pmatrix} 1 & x & y \\ 0 & 1 & -x^\sigma \\ 0 & 0 & 1 \end{pmatrix},$$

where $k, x, y \in F(r)^*$ and $x^\sigma x + y^\sigma + y = 0$.

Then the two-point stabiliser $D = G_{\langle f \rangle \langle e \rangle}$ consists of the group of h_k, and we put L equal to the group of $t_{x,y}$. We have $D \cap L = 1$, $H = DL$ with $|H| = (q^2 - 1)q^3$, and L a normal subgroup, regular on the isotropic points $\langle e + xw + yf \rangle$ other than $\langle f \rangle$ (such points necessitate $x^\sigma x + y^\sigma + y = 0$). A straightforward calculation shows $t_{x,y}$ centralised by h_k if and only if $k^3 = 1$, so that there exists a fixed-point-free automorphism of L.

Lemma 7.32 *The groups $G = SU_3(q)$ are of Suzuki type.*

PROOF. We have G doubly transitive on Ω, where $|\Omega| = q^3 + 1 > 9$. Also, G has no regular normal subgroup, since $G/Z(G)$ is simple, while the point stabiliser H contains the regular normal subgroup L, and $|L| = q^f$ for some $q^f > 8$. This L is a characteristic subgroup of H since $(d, q) = 1$, where $d = |D| = (q + 1)(q - 1)$.

For $q > 2$, the map $\mathcal{N} : F^* \longrightarrow F_0^*$ given by $b \mapsto b^\sigma b$ is called the *norm* epimorphism. Here $b^\sigma b \in F_0^*$ because F_0 is the field of elements fixed by σ (by definition). Put $t \in G \setminus H$ equal to the involution

$$t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

conjugating h_k to $h_{k^{-\sigma}}$. Then $C_t(D) = T = \{h_k : k \in \operatorname{Ker}\mathcal{N}\}$, and $S = \{h_k : k \in F_0^*\}$. Also $|S| = q - 1, |T| = q + 1$ and it is easily shown that if $X \leq D$ is such that $C_L(X) \neq 1$, then $X = T$ and $C_L(T) = \langle t_{0,y} : y \in F\rangle$.

We now turn to the FH-modules M such that $M \cong M^t$ as FD-modules. These are lifted from $H/TL \cong S$ to H. We first consider $p \mid q^2 - q + 1$ such that $p \nmid |G|/(q^2 - q + 1)$. Take $F = k$ in the above, so that M affords a linear k-representation λ, and there exists some linear R-representation μ of H such that $\mu^\pi = \lambda$, by 7.2. Now μ^G appears as the sum of $\chi_t^{(u)}, \chi_{qt}^{(u)}$ in [34], $\chi_t^{(u)}(1) = q^2 - q + 1$, $\chi_{qt}^{(u)}(1) = q(q^2 - q + 1)$.

Lemma 7.33 *Suppose $p \mid q^2 - q + 1$. Then:*

(*i*) $p \nmid q$ *and* $p \nmid q - 1$; *and*
(*ii*) $p = 3$ *if and only if* $p \mid q + 1$;

PROOF. If $p \mid q$ or $q-1$, then $p \mid q(q-1)$, so that $p \mid q^2-q+1$ implies $p = 1$. If $p \mid q + 1$, then $p \nmid q$, and $p \mid q(q - 2) + q + 1$ implies $p \mid q - 2$. Therefore, $p \mid q + 1 - 3$ and we must have $p = 3$.

By the above, if $p \neq 3$ and $p \mid q^2 - q + 1$, then $p \nmid |H|$ or $q + 1$ and χ_1, χ_2 are contained in a block of defect 0. In this case:

Lemma 7.34 *If $p \mid q^2 - q + 1$ and $p \neq 3$, then $M^G = U_1 \oplus U_2$, with U_1, U_2 irreducible. Here $dim(U_1) = q^2 - q + 1$ and $dim(U_2) = q(q^2 - q + 1)$.*

Clearly $p \nmid |\Omega|$ precisely when $p \mid |S| = q - 1$, in which case $Q \in Syl_p(D)$ is a p-Sylow subgroup of G. We may then apply 6.8 to deduce:

Lemma 7.35 *If $p \neq 2$ and $p \mid q-1$, then $M^G = U_1 \oplus U_2$, with U_1, U_2 irreducible. Here $dim(U_1) = q^2 - q + 1$ and $dim(U_2) = q(q^2 - q + 1)$.*

We have not yet been able to complete the case $p \mid q+1$ (and $p \nmid q$), but postulate the following conjecture which has been verified for all prime powers $q \leq 11$ using GAP [20] (see §51 [19] for more information). In particular, $p \mid |\Omega|$ so that M^G is indecomposable by 6.6, and that this case includes $p = 2$ (for which $p \mid q-1$ also).

Conjecture 7.36 *If $p \mid q+1$ then M^G is indecomposable, $M^G/rad(M^G) \cong soc(M^G)$ is irreducible of dimension $q^2 - q + 1$, and $rad(M^G)/soc(M^G)$ irreducible of dimension $(q-1)(q^2 - q + 1)$.*

7.6 Symplectic groups, $Sp(2m, 2)$

Let V be a vector space of dimension $2m$ over a field F with an *alternating form* β, that is, $\beta(u, u) = 0$ for all $u \in V$. Let $\Phi(\beta)$ be the set of all quadratic froms that *polarise to* β, that is,

$$\beta(u, v) = Q(u + v) - Q(u) - Q(v),$$

for all $u, v \in V$. We will assume $F = GF(q)$ with q even, so that there are q^{2m} forms polarising to β.

If we let $f \in GL(V)$ act on $\theta \in \Phi(\beta)$ via $f \cdot \theta(u) := \theta(f^{-1}(u))$ (for all $u \in V$), then $f \cdot \theta \in \Phi(\beta)$ if and only if $f \in Sp(V)$, since $f \cdot \theta$ polarises to $\beta(f^{-1}(-), f^{-1}(-))$. This means $Sp(V)$ acts naturally on $\Phi(\beta)$. However, the action cannot be transitive as $\Phi(\beta)$ contains quadratic forms of types $\epsilon = \pm 1$ (see p.139 [40]), depending on whether the dimension of a maximal totally isotropic subspace is m or $m-1$, respectively.

If $Q \in \Phi(\beta)$ is of type ϵ, then $Sp(V)_Q \cong O^\epsilon(V)$ (the orthogonal group of type ϵ). Thus $Sp(V)$ has 2 orbits $\Phi^\epsilon(\beta)$ of sizes $\frac{q^m}{2}(q^m + \epsilon)$ (see p.70 and p.141 [40]). We assume $q = 2$, in which case it can be shown $O^\epsilon(V)$ is transitive on the sets $\Phi^\epsilon(\beta) \setminus \{Q\}, \Phi^{-\epsilon}(\beta)$, where Q is assumed to be of type ϵ. It follows that $G = Sp_{2m}(2)$ has a doubly transitive action on each of the two sets $\Phi^\epsilon(\beta)$ with cardinalities $2^{m-1}(2^m + \epsilon)$. Further information on these two actions may be found in [25].

Let Ω^ϵ be the commutator subgroup of $Sp(V)_Q = O^\epsilon$. Writing $[V, f] := \mathrm{Im}(1 - f)$ for $f \in O(V)$, define $D(f) := dim[V, f] \pmod 2$ to be the *Dickson invariant*, a homomorphism $D : O(V) \to GF(2)$ (see Theorem (2)(*ii*) in [18], or 11.43 in [40]).

If q is even we *define* $SO(W) := \mathrm{Ker}\, D$, implying $|O(W) : SO(W)| = 2$ if the form of the orthogonal geometry is non-degenerate. For $dim(W) \geq 3$ and $O(W) \neq O^+(4,2)$, we have $SO(W)' = \Omega(W)$ (11.45 [40]). Furthermore, if q is even then $SO(W) = \Omega(W)$ (11.51 in [40]). So if $W = V$ of dimension $2m$ over $GF(2)$, $O^\epsilon(V)$ has 1 non-trivial linear character, λ^ϵ say, for which $|H : \mathrm{Ker}\,\lambda| = 2$.

Definition 7.37 *Let $K \leq H \leq G$ be groups such that $K^x \cap H \leq K$ for all $x \in G$. Then K is said to be strongly closed in H (see §6 [39]).*

We return to our doubly transitive setting, assuming G, H, λ as in Hypothesis 5.1.

Lemma 7.38 *Suppose $|H : \mathrm{Ker}\,\lambda| = 2$. Then λ^G is reducible if and only if* $\mathrm{Ker}\,\lambda$ *is strongly closed in H.*

PROOF. Since $|H : \mathrm{Ker}\,\lambda| = 2$, $\lambda = \pm 1$ and so by Corollary 5.12, λ^G is reducible if and only if $\lambda^t(g) = \lambda(g^t) = 1$ whenever $g \in \mathrm{Ker}\,\lambda$. Since $\mathrm{Ker}\,\lambda \trianglelefteq H$ and $G = H1H \cup HtH$, this is precisely the definition of strong closure of $\mathrm{Ker}\,\lambda$ in H.

By the following, $SO(V)$ is strongly closed in H.

Lemma 7.39 *For all $f \in Sp(V)$ we have $f^{-1}SO(V)f \cap O(V) \leq SO(V)$.*

PROOF. Let $g \in f^{-1}SO(V)f \cap O(V)$ be some element $g = f^{-1}hf$, for $f \in Sp(V)$ and $h \in SO(V)$. Then

$$\begin{aligned} dim[V, f^{-1}hf] &= dim(V^{f^{-1}(1-h)f}) \\ &= dim(V^{1-h}) \\ &= dim[V, h], \end{aligned}$$

by the fact that f is invertible. By definition, $SO(V) := \mathrm{Ker}\,D$ with $D(f) := dim[V, f] \pmod 2$. Therefore $g = f^{-1}hf \in SO(V)$ by the above, since $h \in SO(V)$.

It now follows, from Corollary 5.12, that if M^ϵ is a non-trivial 1-dimensional FO^ϵ-module (where $char(F) \neq 2$), then $(M^\epsilon)^{Sp_{2m}(2)}$ is reducible. When F has characteristic 0 or $char(F) \nmid |Sp_{2m}(2)|$, the degrees of the irreducible constituents can be determined from 6.4 in [39].

Finally, the group $G = Sp_6(2)$ is the only occurence of G with non-trivial Schur multiplier, affording the perfect central extension $2.Sp_6(2)$.

G	Degree n	H	Transitivity
M_{10}	10	$A_6 \cdot 2$	3
M_{11}	11	M_{10}	4
M_{12}	12 (twice)	M_{11}	5
M_{22}	22	$PSL_3(4)$	3
M_{23}	23	M_{22}	4
M_{24}	24	M_{23}	5

Table 7.12
The Mathieu groups and their actions.

Lemma 7.40 *Let $G = Sp_6(2)$ and H^ϵ be the point stabiliser in its doubly transitive action, as above. Let $\overline{H^\epsilon}$ be the pre-image of H^ϵ in $2.Sp_6(2)$. Then all linear characters of $\overline{H^\epsilon}$ are lifted from H^ϵ to $\overline{H^\epsilon}$.*

In view of Table 2.6, we conclude this section with a result which follows by inspection of the modular atlas [24] or by using Gap [20]:

Lemma 7.41 *Let F be a field of characteristic $p \neq 2$, and M the non-trivial 1-dimensional module for the point stabiliser $H^- \leq Sp_6(2)$. Then $M^G = U_1 \oplus U_2$ with U_1, U_2 irreducible of dimensions $7, 21$ respectively.*

7.7 The Mathieu groups

Among the Mathieu groups we find the only finite 4- and 5-transitive groups which are not alternating or symmetric. The relationship between the Mathieu groups as point stabilisers of each other can be seen in Table 7.12. Note that M_{11} has a second doubly transitive action, one on 12 points, which we will deal with seperately in §7.8.

The faithful minimal submodules of $F\Omega$ for M_{22}, M_{23} and $F = GF(2)$ are described in §8 [23].

Let G be a simple Mathieu group or a finite central extension thereof. Then G has a non-perfect point stabiliser if and only if $G = M_{11}$ with point stabiliser $A_6 \cdot 2$, or $G = 2.M_{12}$ with point stabiliser $2 \times M_{11}$. In both cases H affords one non-trivial linear K-character λ and λ^G is irreducible.

Lemma 7.42 *The linear representations λ above exist for $p \neq 2$, and λ^G is reducible if and only if $p = 3$, in which case λ^G affords the Brauer character:*

(*i*) $5a + 1a + 5b$ *for* $G = M_{11}$*; or*
(*ii*) $6a + 6b$ *for* $G = 2.M_{12}$.

In both cases the non-trivial characters are duals of each other, and they may be realised over $GF(3)$.

Lemma 7.43 *With $F = GF(3)$, suppose the FH-module M affords λ, as in 7.42. Then M^G contains a unique irreducible submodule of dimension $5, 6$, given that $G = M_{11}, 2.M_{12}$ respectively.*

7.8 The groups M_{11} and $PSL_2(11)$

The group $G = M_{11}$ has a 3-transitive action on 12 points with point stabiliser $PSL_2(11)$, which is thereby 2-transitive on the remaining 11 points. For a construction of these group actions see example 7.5.2 in [16] or §3.7 in [3]. Since the 2-point stabiliser of M_{11} acting on 12 points is isomorphic to A_5, the 1- and 2-point stabilisers of G are both simple, and they afford no non-trivial linear representations.

Lemma 7.44 *Let $\mathcal{D}$ be the heart of $PSL_2(11)$ acting on 11 points, and suppose $p = 3$. Then $\mathcal{D}$ contains a unique 5-dimensional irreducible FG-module U, and $soc(F\Omega) = \mathcal{C} \oplus U$.*

7.9 The group $P\Sigma L_2(8)$

Recall that $G = P\Sigma L_2(8)$ is the only faithful finite doubly transitive group whose socle is primitive and simple, but not doubly transitive; $P\Sigma L_2(8) \cong R(3)$ in its natural doubly transitive action on 28 points, and $PSL_2(8) \cong R(3)'$.

Lemma 7.45 *Let $G = P\Sigma L_2(8)$ act doubly transitively on a set Ω of 28 points with H a point stabiliser. Then H affords 6 linear representations, 3 of which extend to G.*

PROOF. As in §7.4, we may write $H = DL$, with D the stabiliser of $\infty, 0 \in \Omega$, and L a regular normal subgroup. Since $G \cong R(3)$, $D \cong \mathbb{Z}_2$ does not induce a fixed-point-free automorphism of L (see §7.4). A straightforward calculation yields $r_{\alpha,0,0} \in H'$ for all $\alpha \in GF(3)$. Furthermore, $\langle r_{1,0,0} \rangle$ is a cyclic subgroup of L of order 9 (containing $\langle r_{0,0,1} \rangle$) trivially intersecting the subgroup $\langle r_{0,1,0} \rangle \cong GF(3)$, and $L/\langle r_{1,0,0} \rangle \cong GF(3)$. We now have $H' = \langle r_{1,0,0} \rangle$ so that H affords 6 linear representations.

Now $G' = PSL_2(8)$ has index 3 in G, and G affords 2 non-trivial linear representations μ, μ^{-1}. By inspection of the ATLAS [13], G' contains no elements of order 6, while H does. Therefore μ_H, μ_H^{-1} are not the trivial representation and 3 of the 6 linear representations of H extend to G.

Lemma 7.46 *Let $\widehat{U_1}, \widehat{U_2}, \widehat{U_3}$ be the (where possible distinct) FG-modules such that $U_i = \widehat{U}_{iH}$ is lifted from $H/(D\langle r_{1,0,0}\rangle)$, for $i = 1, 2, 3$. Furthermore, if p is odd let M be the non-trivial 1-dimensional FH-module lifted from $H/L \cong \mathbb{Z}_2$. Then for $i = 1, 2, 3$:*

(*i*) *$U_i^G \cong \widehat{U_i} \otimes K\Omega$ if $p = 2, 7$, with $soc(U_i^G) \cong \widehat{U_i}$;*
(*ii*) *U_i^G is completely reducible as $\widehat{U_i} \oplus (\widehat{U_i} \otimes \mathcal{C}^{\perp})$ if $p \neq 2, 7$; and*
(*iii*) *$(U_i \otimes M)^G \cong \widehat{U_i} \otimes M^G$ is completely reducible as $V_{i1} \oplus V_{i2}$ if p is odd, for some V_{i1}, V_{i2} of dimensions $7, 21$ respectively.*

7.10 The group A_7

The 3-dimensional projective space $PG_3(2)$ has 15 points and admits the automorphism group $PGL_4(2)$. Since $A_8 \cong PSL_4(2)$, there exists a subgroup of $PSL_4(2)$ of index 8 and isomorphic to A_7. This group happens to act 2-transitively on $PG_3(2)$.

When $p = 2$, the heart $\mathcal{D}$ over $F = GF(2)$ of G acting on Ω contains a unique 4-dimensional irreducible submodule U, such that $\mathcal{D}/rad\mathcal{D} \cong U^*$, in agreement with 7.10. Moreover, between the composition factors U, U^* sits a 6-dimensional composition factor.

Lemma 7.47 *Let G be A_7 or some perfect central extension thereof, with H a point stabiliser in the doubly transitive action of G. Then H affords a non-trivial linear representation if and only if $G = 3.A_7$, in which case it affords two non-trivial linear representations λ, λ^{-1}.*

Lemma 7.48 *Let $G = 3.A_7$ and λ, λ^{-1} be the K-characters of 7.47. Then $\lambda^G \neq (\lambda^{-1})^G$ are irreducible.*

PROOF. By inspection of the ATLAS [13], 7.3 provides two possibilities for the induced characters, namely $15b, 15c$. Since $PSL_2(7)$ contains no elements of order 6, the characters must take on the value 0 on the elements of the conjugacy class labelled $6A$, and only $15c$ has this property. By expansion of the character table (to include the second cohort of $3.A_7$) we obtain two distinct irreducible characters $15c, \overline{15c}$ of G.

Suppose $p \mid |G|$. Then the linear representations λ, λ^{-1} of 7.47 exist over F provided $F \geq GF(4), GF(25)$ or $GF(7)$. By inspection of the modular atlas

[24] we have λ^G reducible if and only if $p = 7$, in which case $\lambda^G, (\lambda^{-1})^G$ afford the Brauer characters $6a + 9a$. Both composition factors may be realised over $GF(7)$, in which case they are duals of each other.

7.11 The Higman-Sims group, HS

The group HS can be described as the automorphism group of a combinatorial geometry consisting of a set Ω of 176 points, and a set Γ of 176 blocks (called 'quadrics'), such that each block contains 50 points and each point is contained in 50 blocks. Each pair of points is incident with exactly 14 blocks giving a $2-(176, 50, 14)$ symmetric design. The group HS acts doubly transitively on Ω as well as on Γ (for a more detailed construction, see 4.5 [3]). These actions are inequivalent but permutation isomorphic by the unique non-inner automorphism of HS.

The heart $\mathcal{D}$ over $F = GF(3)$ of G in either action affords the Brauer character $49a + 77a + 49b$, with $soc(\mathcal{D})$ irreducible of dimension 49, and $\mathcal{D}/rad(\mathcal{D}) \cong soc(\mathcal{D})^*$.

Lemma 7.49 *Let $G = HS$, acting doubly transitively on Ω as above. Then a point stabiliser H_1 affords precisely one non-trivial linear representation λ_1. The permutation isomorphic action of G on Γ yields another point stabiliser H_2 with non-trivial linear representation λ_2, and if $F = K$ we have*

$$\begin{aligned}\lambda_1^G &= 22a + 154b\\ \lambda_2^G &= 22a + 154c.\end{aligned}$$

Furthermore, $\operatorname{Im}\lambda_i = \pm 1$ *for $i = 1, 2$.*

Lemma 7.50 *Assume $p \neq 2$ and let M_i be a 1-dimensional FH_i-module affording the linear representation λ_i of 7.49 ($i = 1, 2$). Then either:*

- *(i) $p \neq 5$ and $M_i^G \cong U \oplus V_i$, where U is irreducible of dimension 22, V_i irreducible of dimension 154 (for $i = 1, 2$); or*
- *(ii) $p = 5$ and $M_i^G/rad(M_i^G) \cong soc(M_i^G)$ is irreducible of dimension 21.*

Moreover, these submodules are realisable over the prime subfield of F.

Finally, we turn to the covering group $2.HS$.

Lemma 7.51 *Let $G = 2.HS$ be the covering group of HS. Then a point stabiliser H of G affords 2 non-trivial linear representations λ such that $Z(G) \not\leq$* $\operatorname{Ker}\lambda$, *and with* $\operatorname{Im}\lambda = \pm 1$. *The λ^G are irreducible as K-characters, and the* *two inequivalent isomorphic doubly transitive actions of G give rise to two distinct such irreducible K-characters.*

PROOF. The group HS, acting doubly transitively on 176 points, has a point stabiliser H isomorphic to $U_3(5){:}2$. By inspection of [13], this group has the Schur multiplier $\mathbb{Z}_2$, and the point stabilisers of $2.HS$ are isomorphic with $2 \times (U_3(5) : 2)$. Therefore, the subgroup $2 \times H$ has a commutator subgroup isomorphic to $U_3(5)$, and so a point stabiliser of $2.HS$ affords 4 linear K-characters. Two such characters are lifted from H to $2.H$, and are described in 7.49. The remaining two can be treated using GAP [20].

We will assume $p \neq 2$, and take H_1, H_2 as before. Let M_i be a 1-dimensional FH_i-module affording λ_i, one of the linear representations of 7.51 ($i = 1, 2$).

Lemma 7.52 *With $p \neq 2$, the FG-module M_i^G is reducible if and only if $p = 5$ ($i = 1, 2$), in which case M_1^G, M_2^G affords the Brauer character $28a+120a+28b$, $28a + 120b + 28b$ respectively. Indeed, with $F = GF(5)$, M_i^G contains a unique 28-dimensional irreducible FG-submodule.*

PROOF. Again, the modules M_i afford the non-trivial λ_i whenever $p \neq 2$, since λ_i takes on the values ± 1 by 7.49 ($i = 1, 2$). An inspection of [24] reveals the decomposition of the Brauer character as described, while and 7.1 guarantees the realisation of the composition factors over $F = GF(5)$. Since M_i^G is self-dual (for $i = 1, 2$), M_i^G must have a unique 28-dimensional irreducible socle ($i = 1, 2$) by 5.21.

7.12 The Conway group, Co_3

Conway's third group Co_3 is a sporadic finite simple group. It has a 2-transitive action on a set Ω of 276 points, with point stabiliser $McL : 2$, the full automorphism group of the McLaughlin group. The group Co_3 also contains a subgroup isomorphic to HS, but this does not concern us here.

A point stabiliser $H \cong McL : 2$ of $G = Co_3$ in its doubly transitive action affords a unique non-trivial linear representation λ, such that λ takes on the values ± 1. Furthermore, as a K-character we have $\lambda^G = 23a + 253b$.

Lemma 7.53 *Assume $p \neq 2$ and let M be an FH-module affording the linear representation λ above. Then either:*

(*i*) *$p = 3$ and $M_i^G/rad(M_i^G) \cong soc(M_i^G)$ is irreducible of dimension 22;*

(*ii*) $p = 5$ *and* $M_i^G/rad(M_i^G) \cong soc(M_i^G)$ *is irreducible of dimension 23; or*

(*iii*) $p \neq 3, 5$ *and* $M^G \cong U \oplus V$, *where* U, V *are irreducible of dimension 23, 253 respectively.*

Moreover, these submodules are realisable over the prime subfield of F.

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